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Interacting partially directed self avoiding walk: scaling limits

Philippe Carmona* Nicolas Pétrélis*

Abstract

This paper is dedicated to the investigation of a $1+1$ dimensional self-interacting and partially directed self-avoiding walk. The intensity of the interaction between monomers is denoted by $\beta \in (0, \infty)$ and there exists a critical threshold β_c which determines the three regimes displayed by the model, i.e., *extended* for $\beta < \beta_c$, *critical* for $\beta = \beta_c$ and *collapsed* for $\beta > \beta_c$.

In [4], physicists displayed some numerical results concerning the typical growth rate of some geometric features of the path as its length L diverges. From this perspective the quantities of interest are the horizontal extension of the path and its lower and upper envelopes.

With the help of a new random walk representation, we proved in [10] that the path grows horizontally like $\sqrt{L}$ in its collapsed regime and that, once rescaled by $\sqrt{L}$ vertically and horizontally, its upper and lower envelopes converge towards some deterministic Wulff shapes.

In the present paper, we bring the geometric investigation of the path several steps further. In the collapsed regime, we identify the joint limiting distribution of the fluctuations of the upper and lower envelopes around their associated limiting Wulff shapes, rescaled in time by $\sqrt{L}$ and in space by $L^{1/4}$. In the critical regime we identify the limiting distribution of the horizontal extension rescaled by $L^{2/3}$ and we show that the excess partition function decays as $L^{2/3}$ with an explicit prefactor. In the extended regime, we prove a law of large number for the horizontal extension of the polymer rescaled by its total length L, we provide a precise asymptotics of the partition function and we show that its lower and upper envelopes, once rescaled in time by L and in space by $\sqrt{L}$, converge towards the same Brownian motion.

Keywords: Polymer collapse; phase transition; Brownian area; large deviations.
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*Laboratoire de Mathématiques Jean Leray UMR 6629, Université de Nantes, France.
E-mail: philippe.carmona@univ-nantes.fr
E-mail: nicolas.petrelis@univ-nantes.fr

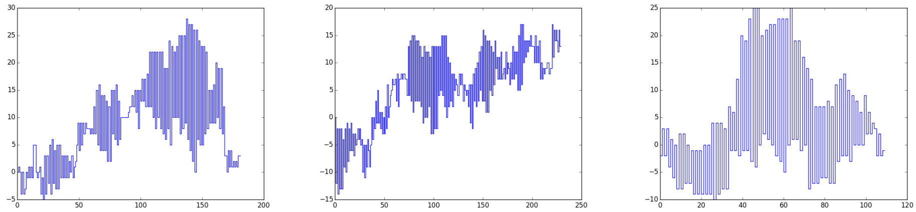

Figure 1: Simulations of IPDSAW for critical temperature $\beta = \beta_c$ and length $L = 1600$

1 Introduction

We consider a model of statistical mechanics introduced in [34] and referred to as interacting partially directed self avoiding walk (IPDSAW). The model is a $(1+1)$-dimensional partially directed version of the interacting self-avoiding walk (ISAW) introduced in [18] as a model for an homopolymer in a poor solvent.

The aim of our paper is to pursue the investigation of the IPDSAW initiated in [29] and [10] and in particular to display the infinite volume limit of some features of the model when the size of the system diverges for each of the three regimes: collapsed, critical and extended. The first object to be considered is the horizontal extension of the path. Then, we will consider the whole path, properly rescaled and look at its infinite volume limit in the extended phase and in the collapsed phase.

Let us point that numerical simulations are difficult (see e.g. [4]) and have not led to theoretical results about the path properties of the polymer in the three regimes that we establish in this paper.

1.1 Model

The model can be defined in a simple manner. An allowed configuration for the polymer is given by a family of oriented vertical stretches. To be more specific, for a polymer made of $L \in \mathbb{N}$ monomers, the possible configurations are gathered in $\Omega_L := \bigcup_{N=1}^{L} \mathcal{L}_{N,L}$, where $\mathcal{L}_{N,L}$ is the set consisting of all families made of N vertical stretches that have a total length $L - N$, that is

$$\mathcal{L}_{N,L} = \Big\{ l \in \mathbb{Z}^N : \textstyle\sum_{n=1}^{N} |l_n| + N = L \Big\}. \tag{1.1}$$

Note that with such configurations, the modulus of a given stretch corresponds to the number of monomers constituting this stretch (and the sign gives the direction upwards or downwards). Moreover, any two consecutive vertical stretches are separated by a monomer placed horizontally and this explains why $\sum_{n=1}^{N} |l_n|$ must equal $L - N$ in order for $l = (l_i)_{i=1}^{N}$ to be associated with a polymer made of L monomers (see Fig. 2).

The repulsion between the monomers and the solvent around them is taken into account in the Hamiltonian associated with each path $l \in \Omega_L$ by rewarding energetically those pairs of consecutive stretches with opposite directions, i.e.,

$$H_{L,\beta}(l_1, \dots, l_N) = \beta \textstyle\sum_{n=1}^{N-1} (l_n \,\widetilde{\wedge}\, l_{n+1}), \tag{1.2}$$

where

$$x \,\widetilde{\wedge}\, y = \begin{cases} |x| \wedge |y| & \text{if } xy < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

One can already note that large Hamiltonians will be assigned to trajectories made of few but long vertical stretches with alternating signs. Such paths will be referred to

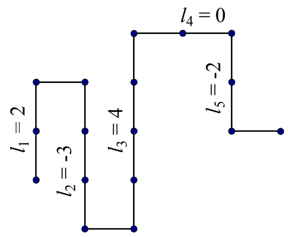

Figure 2: Example of a trajectory $l \in \mathcal{L}_{N,L}$ with $N = 5$ vertical stretches, a total length $L = 16$ and an Hamiltonian $H_{L,\beta}(\pi) = 5\beta$.

as collapsed configurations. With the Hamiltonian in hand we can define the polymer measure as

$$P_{L,\beta}(l) = \frac{e^{H_{L,\beta}(l)}}{Z_{L,\beta}}, \quad l \in \Omega_L, \tag{1.4}$$

where $Z_{L,\beta}$ is the partition function of the model, i.e.,

$$Z_{L,\beta} = \sum_{N=1}^{L} \sum_{l \in \mathcal{L}_{N,L}} e^{H_{L,\beta}(l)}. \tag{1.5}$$

1.2 Random walk representation and collapse transition

An alternative probabilistic representation of the partition function has been introduced in [29]. For $\beta > 0$ we introduce an auxiliary random walk $V = (V_i)_{i=0}^{\infty}$ of law $\mathbf{P}_\beta$, starting from 0, and whose increments $(U_i)_{i=0}^{\infty}$ are i.i.d. and follow a discrete Laplace distribution, i.e.,

$$\mathbf{P}_\beta(U_1 = k) = \frac{e^{-\frac{\beta}{2}|k|}}{c_\beta} \quad \forall k \in \mathbb{Z} \quad \text{with} \quad c_\beta := \frac{1+e^{-\beta/2}}{1-e^{-\beta/2}}. \tag{1.6}$$

Such walk allows us to provide an alternative expression of the partition function, i.e.,

$$\widetilde{Z}_{L,\beta} = e^{-\beta L} Z_{L,\beta} = c_\beta \sum_{N=1}^{L} (\Gamma_\beta)^N \mathbf{P}_\beta(V \in \mathcal{V}_{N,L-N}), \tag{1.7}$$

where $\mathcal{V}_{N,L-N} := \{V : G_N(V) = L - N, V_{N+1} = 0\}$, where $G_N(V) := \sum_{i=1}^{N} |V_i|$ is the geometric area in between V and the horizontal axis up to time N and where

$$\Gamma_\beta = \frac{c_\beta}{e^\beta} \quad \text{and} \quad c_\beta := \frac{1+e^{-\beta/2}}{1-e^{-\beta/2}}. \tag{1.8}$$

For later use, we also introduce $A_N(V) := \sum_{i=1}^{N} V_i$ the algebraic counterpart of $G_N(V)$. We will recall briefly in Section 3.1 how (1.7) can be obtained, but let us observe already that the *excess free energy* defined as

$$\widetilde{f}(\beta) := \lim_{L\to\infty} \frac{1}{L} \log \widetilde{Z}_{L,\beta} \tag{1.9}$$

loses its analyticity at β_c, the unique solution of $\Gamma_\beta = 1$. For $\beta \geq \beta_c$ the inequality $\Gamma_\beta \leq 1$ indeed yields that $\widetilde{f}(\beta) = 0$ since those terms indexed by $N \sim \sqrt{L}$ in (1.7) decay subexponentially. As a consequence the trajectories dominating $\widetilde{Z}_{L,\beta}$ have a small horizontal extension, i.e., $N = o(L)$. When $\beta < \beta_c$ in turn, $\Gamma_\beta > 1$ and since for $c \in (0, 1]$ the quantity $P_\beta(\mathcal{V}_{cL,(1-c)L})$ decays exponentially fast with a rate that vanishes as $c \to 0$

we can claim that the dominating trajectories in $\widetilde{Z}_{L,\beta}$ have an horizontal extension of order L, and moreover that $\widetilde{f}(\beta) > 0$. The phase diagram $[0, \infty)$ is therefore partitioned into a collapsed phase denoted by $\mathcal{C}$ and an extended phase denoted by $\mathcal{E}$, i.e,

$$\mathcal{C} := \{\beta : \widetilde{f}(\beta) = 0\} = \{\beta : \beta \geq \beta_c\}, \tag{1.10}$$
$$\mathcal{E} := \{\beta : \widetilde{f}(\beta) > 0\} = \{\beta : \beta < \beta_c\}.$$

We shall see that, in fact, there are three regimes; collapsed ($\beta > \beta_c$), critical ($\beta = \beta_c$) and extended ($\beta < \beta_c$), in which the asymptotics of the partition function and the path properties are radically different.

2 Main results

We observe that the definition of the polymer measure in (1.4) is left unchanged if we replace the denominator by $\widetilde{Z}_{L,\beta}$ (cf (1.7)) and substract $L\beta$ to the Hamiltonian.

2.1 Scaling limit of the horizontal extension

Displaying sharp asymptotic estimates of the partition function as the system size diverges is a major issue in statistical mechanics. Computing the probability mass of a certain subset of trajectories under the polymer measure indeed requires to have a good control on the denominator in (1.4). For the extended and the critical regimes, we display in Theorem 2.1 below an equivalent of the partition function allowing us e.g to exhibit the polynomial decay rate of the partition function at the critical point. For the collapsed regime, in turn, we recall the bounds on $\widetilde{Z}_{L,\beta}$ that had been obtained in [10] allowing us to identify its sub-exponential decay rate.

Note that in Remark 2.3 below, we provide some complements concerning Theorems 2.1 and 2.2 among which the exact value of some pre-factors when an expression is available. We also denote by f_{ex} the density of the area below a normalized Brownian excursion (see e.g. [25]) and we set $C_\beta := (\mathbf{E}_\beta(V_1^2))^{-1/2}$. Thus, we can define $w(x) = C_\beta\, f_{ex}(C_\beta\, x)$. We recall the definition of $\widetilde{f}(\beta)$ in (1.9).

Theorem 2.1 (Asymptotics of the partition function)**.** *(1) For $\beta < \beta_c$, there exists a $c > 0$ such that*

$$\widetilde{Z}_{L,\beta} = c\, e^{\widetilde{f}(\beta) L}(1 + o(1)),$$

(2) for $\beta = \beta_c$, there exists a $c > 0$ such that

$$\widetilde{Z}_{L,\beta} = \frac{c}{L^{2/3}}(1 + o(1)) \quad \textit{with} \quad c = \frac{1 + e^{-\frac{\beta}{2}}}{(24\,\pi\,\mathbf{E}_\beta(V_1^2))^{\frac{1}{2}} \int_0^{+\infty} x^{-3} w(x^{-\frac{3}{2}})\, dx},$$

(3) for $\beta > \beta_c$, there exists a unique real number $m(\beta) > 0$ and $c_1, c_2, \kappa > 0$ such that

$$\frac{c_1}{L^\kappa} e^{-m(\beta)\sqrt{L}} \leq \widetilde{Z}_{L,\beta} \leq \frac{c_2}{\sqrt{L}} e^{-m(\beta)\sqrt{L}} \quad \textit{for} \quad L \in \mathbb{N}.$$

For each $l \in \Omega_L$, the variable N_l denotes the horizontal extension of l, i.e., the integer $N \in \{1, \dots, L\}$ such that $l \in \mathcal{L}_{N,L}$. Theorem 2.2 below gives the scaling limit of the horizontal extension of a typical path l sampled from $P_{L,\beta}$ and as $L \to \infty$ (for the sake of completeness, we again integrate the collapsed regime into the theorem although this regime was dealt with in [10, Theorem D]).

Theorem 2.2 (Horizontal extension)**.** *(1) if $\beta < \beta_c$, there exists a real constant $e(\beta) \in (0, 1)$ such that*

$$\lim_{L\to\infty} P_{L,\beta}\Big(\Big|\frac{N_l}{L} - e(\beta)\Big| \geq \varepsilon\Big) = 0. \tag{2.1}$$

(2) if $\beta = \beta_c$, then

$$\lim_{L\to\infty} \frac{N_l}{L^{2/3}} =_{law} C_\beta^{2/3} g_1,$$

where $g_a = \inf\left\{t > 0 \int_0^t |B_s|\, ds = a\right\}$ is the continuous inverse of the geometric Brownian area, and we consider g_1 under the conditional law of the Brownian motion conditioned by $B_{g_1} = 0$.

(3) If $\beta > \beta_c$, there exists a unique real number $a(\beta) > 0$ such that

$$\lim_{L\to\infty} P_{L,\beta}\Big(\Big|\frac{N_l}{\sqrt{L}} - a(\beta)\Big| \geq \varepsilon\Big) = 0. \tag{2.2}$$

Remark 2.3.

(1) For the extended regime, in Section 6, we will decompose each path into a succession of patterns (sub pieces) and we will associate with our model an underlying regenerative process $(\sigma_i, \nu_i, y_i)_{i\in\mathbb{N}}$ of law $\mathfrak{P}_\beta$ in such a way that σ_i (resp. ν_i, resp. y_i) plays the role of the number of monomers constituting the ith pattern (resp. the horizontal extension of the ith pattern, resp. the vertical displacement of the ith pattern). Then, the constant c in Theorem 2.1 (1) and the limiting rescaled horizontal extension in Theorem 2.2 (1) satisfy

$$c = \tfrac{1}{\mathfrak{E}_\beta(\sigma_1)} \quad \text{and} \quad e(\beta) = \tfrac{\mathfrak{E}_\beta(\nu_1)}{\mathfrak{E}_\beta(\sigma_1)}.$$

(2) For the critical regime $\beta = \beta_c$, the appearance of the distribution of g_1 is explained at the end of Section 4.

2.2 Scaling limit of the vertical extension

The fact that each trajectory $l \in \Omega_L$ is made of a succession of vertical stretches makes it convenient to give a representation of the trajectory in terms of its upper and lower envelopes. Thus, we pick $l \in \mathcal{L}_{N,L}$ and we let $\mathcal{E}_l^+ = (\mathcal{E}_{l,i}^+)_{i=0}^{N+1}$ and $\mathcal{E}_l^- = (\mathcal{E}_{l,i}^-)_{i=0}^{N+1}$ be the upper and the lower envelopes of l, i.e., the $(1+N)$-step paths that link the top and the bottom of each stretch consecutively. Thus, $\mathcal{E}_{l,0}^+ = \mathcal{E}_{l,0}^- = 0$,

$$\mathcal{E}_{l,i}^+ = \max\{l_1 + \cdots + l_{i-1}, l_1 + \cdots + l_i\}, \quad i \in \{1, \dots, N\}, \tag{2.3}$$

$$\mathcal{E}_{l,i}^- = \min\{l_1 + \cdots + l_{i-1}, l_1 + \cdots + l_i\}, \quad i \in \{1, \dots, N\}, \tag{2.4}$$

and $\mathcal{E}_{l,N+1}^+ = \mathcal{E}_{l,N+1}^- = l_1 + \cdots + l_N$ (see Fig. 3). Note that the area in between these two envelopes is completely filled by the path and therefore, we will focus on the scaling limits of $\mathcal{E}_l^+$ and $\mathcal{E}_l^-$.

At this stage, we define $\widetilde{Y} : [0,1] \to \mathbb{R}$ to be the time-space rescaled cadlag process of a given $(Y_i)_{i=0}^{N+1} \in \mathbb{Z}^{N+1}$ satisfying $Y_0 = 0$. Thus,

$$\widetilde{Y}(t) = \frac{1}{N+1} Y_{\lfloor t\,(N+1)\rfloor}, \quad t \in [0,1], \tag{2.5}$$

and for each $l \in \mathcal{L}_{N,L}$ we let $\widetilde{\mathcal{E}}_l^+$, $\widetilde{\mathcal{E}}_l^-$ be the time-space rescaled processes associated with the upper envelope $\mathcal{E}_l^+$ and with the lower envelope $\mathcal{E}_l^-$, respectively.

In this paper we will focus on the infinite volume limit of the whole path inside the collapsed phase ($\beta > \beta_c$) and in the extended phase ($\beta < \beta_c$). Concerning the critical regime ($\beta = \beta_c$) this limit will be discussed as an open problem in Section 2.3 below.

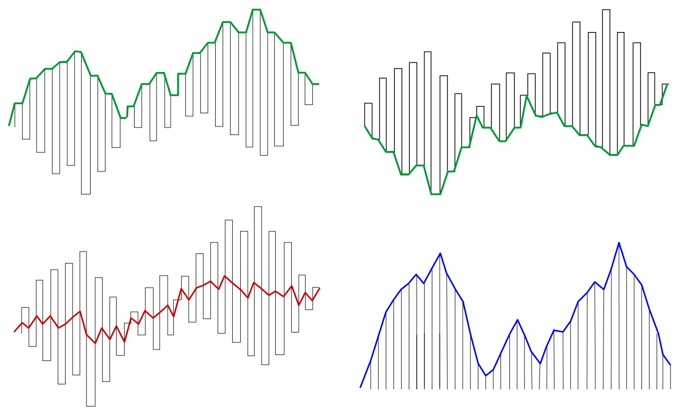

Figure 3: Example of the upper envelope (top-left picture), of the lower envelope (top-right picture), of the middle line (bottom left picture) and of the profile (bottom right picture) of a given trajectory (in dashed line).

The collapsed phase ($\beta > \beta_c$)

The collapsed regime was studied in [10], where a particular decomposition of the path into *beads* has been introduced. A bead is a succession of non-zero vertical stretches with alternating signs which ends when two consecutive stretches have the same sign (or when a stretch is null). Such a decomposition is meaningful geometrically and we proved in [10, Theorem C] that there is a unique macroscopic bead in the collapsed regime and that the number of monomers outside this bead are at most of order $(\log L)^4$.

The next step, in the geometric description of the path, consisted in determining the limiting shapes of the envelopes of this unique bead. This has been achieved in [10] where the rescaled upper envelope (respectively lower envelope) is shown to converge in probability towards a *deterministic Wulff shape* γ^*_β (resp. $-\gamma^*_\beta$) defined as follows

$$\gamma^*_\beta(s) = \int_0^s \mathfrak{L}'\big[(\tfrac{1}{2} - x)\widetilde{h}_0(\tfrac{1}{a(\beta)^2}, 0)\big]\,dx, \quad s \in [0,1], \tag{2.6}$$

where $\mathfrak{L}$ is defined in (5.3) and $\widetilde{h}_0$ in (5.10). Thus, we obtained

Theorem 2.4. ([10] Theorem E) *For $\beta > \beta_c$ and $\varepsilon > 0$,*

$$\begin{aligned} &\lim_{L\to\infty} P_{L,\beta}\Big(\|\widetilde{\mathcal{E}}^+_l - \frac{\gamma^*_\beta}{2}\|_\infty > \varepsilon\Big) = 0, \\ &\lim_{L\to\infty} P_{L,\beta}\Big(\|\widetilde{\mathcal{E}}^-_l + \frac{\gamma^*_\beta}{2}\|_\infty > \varepsilon\Big) = 0. \end{aligned} \tag{2.7}$$

This Theorem has also been stated as a *Shape Theorem* in [10]. The natural question that comes to mind is: are we able to identify the fluctuations around this shape? For technical reasons that will be discussed in Remark 2.6 below, we are not able to identify such a limiting distribution. However, we can prove a close convergence result by working on a particular mixture of those measures $P_{L',\beta}$ for $L' \in K_L := L + [-\varepsilon(L), \varepsilon(L)] \cap \mathbb{N}$ with $\varepsilon(L) := (\log L)^6$. Thus, we define the extended set of trajectories $\widetilde{\Omega}_L = \cup_{L'\in K_L} \Omega_{L'}$, and we let $\widetilde{P}_{L,\beta}$ be a mixture of those $\{P_{L',\beta},\, L' \in K_L\}$ defined by

$$\widetilde{P}_{L,\beta}(l) := \sum_{L'\in K_L} \frac{\widetilde{Z}_{L',\beta}}{\sum_{k\in K_L} \widetilde{Z}_{k,\beta}} P_{L',\beta}(l)\, 1_{\{l\in\Omega_{L'}\}}, \quad \text{for } l \in \widetilde{\Omega}_L. \tag{2.8}$$

In other words, $\widetilde{P}_{L,\beta}$ satisfies

$$\widetilde{P}_{L,\beta}\big(\,\cdot\,|\,\Omega_{L'}\big) = P_{L',\beta}(\cdot) \quad \text{and} \quad \widetilde{P}_{L,\beta}\big(\Omega_{L'}\big) = \frac{\widetilde{Z}_{L',\beta}}{\sum_{k\in K_L} \widetilde{Z}_{k,\beta}}, \quad \text{for } L' \in K_L. \tag{2.9}$$

We denote by $\widetilde{Q}_{L,\beta}$ the law of the fluctuations of the envelopes around their limiting shapes, that is the law of the random processes

$$\sqrt{N_l}\left(\widetilde{\mathcal{E}}_l^{+}(s) - \tfrac{\gamma_\beta^*(s)}{2}, \widetilde{\mathcal{E}}_l^{-}(s) + \tfrac{\gamma_\beta^*(s)}{2}\right)_{s\in[0,1]}, \tag{2.10}$$

with l sampled from $\widetilde{P}_{L,\beta}$. Finally, let us note that stating Theorem 2.5 requires using function $\widetilde{H}$ that will be defined in Section 5.1. We obtain the following limit.

Theorem 2.5 (Fluctuations of the convex envelopes around the Wulff shape)**.** *For $\beta > \beta_c$, and $H = \widetilde{H}(q_\beta, 0), q_\beta = \frac{1}{a(\beta)^2}$ we have the convergence in distribution*

$$\widetilde{Q}_{L,\beta} \xrightarrow[L\to\infty]{d} \Big(\xi_H + \frac{\xi_H^c}{2}, \xi_H - \frac{\xi_H^c}{2}\Big), \tag{2.11}$$

where for $H = (h_0, h_1)$ such that $[h_0, h_0 + h_1] \subset D = (-\beta/2, \beta/2)$, the process $\xi_H = (\xi_H(t), 0 \le t \le 1)$ is centered and Gaussian with covariance

$$\mathbb{E}\left[\xi_H(s)\, \xi_H(t)\right] = \int_0^{s\wedge t} \mathfrak{L}''((1-x)h_0 + h_1)\, dx,$$

and where $\xi_H^c := (\xi_H^c(t), 0 \le t \le 1)$ is a process independent of ξ_H which has the law of ξ_H conditionally on $\xi_H(1) = \int_0^1 \xi_H(s)\, ds = 0$.

From Theorem 2.5 we deduce that the fluctuations of both envelopes around their limiting shapes are of order $L^{1/4}$.

Remark 2.6. The reason why we prove Theorem 2.5 under the mixture $\widetilde{P}_{L,\beta}$ rather than $P_{L,\beta}$ is the following. We need to establish a local limit theorem for the associated random walk of law $\mathbf{P}_\beta$ conditioned on having a large geometric area $G_N(V)$ and we are unable to do it. Fortunately, we know how to condition the random walk on having a large algebraic area $A_N(V)$ and under the mixture $\widetilde{P}_{L,\beta}$ we are able to compare quantitatively these two conditionings (see Step 2 of the proof of Proposition 5.2 in Section 5.5).

In the construction of the mixture law $\widetilde{P}_{L,\beta}$ (cf. (2.8)), the choice of the prefactors of those $P_{L',\beta}$ with $L' \in K_L$ may appear artificial. However it is conjectured (see e.g. [20, Section 8]) that our inequalities in Theorem 2.1 (3) can be improved to

$$\widetilde{Z}_{L,\beta} \sim \frac{B}{L^{3/4}} e^{-m(\beta)\sqrt{L}} \quad \text{with } B > 0,$$

so that the ratio of any two prefactors would converges to 1 as $L \to \infty$ uniformly on the choice of the indices of the two prefactors in K_L. In other word, $\widetilde{P}_{L,\beta}$ should, in first approximation, be the uniform mixture of those $\{P_{L',\beta},\, L' \in K_L\}$.

Remark 2.7. We observe that one can recover the envelopes from two auxiliary processes, i.e, the *middle line* M_l and the *profile* $|l|$. Thus, we associate with each $l \in \mathcal{L}_{N,L}$ the path $|l| = (|l_i|)_{i=0}^{N+1}$ (with $l_{N+1} = 0$ by convention) and the path $M_l = \big(M_{l,i}\big)_{i=0}^{N+1}$ that links the middles of each stretch consecutively, i.e., $M_{l,0} = 0$ and

$$M_{l,i} = l_1 + \cdots + l_{i-1} + \frac{l_i}{2}, \quad i \in \{1, \ldots, N\}, \tag{2.12}$$

and $M_{l,N+1} = l_1 + \cdots + l_N$ (see Fig. 3). With the help of (2.5), we let $\widetilde{M}_l$ and $\widetilde{l}$ be the time-space rescaled processes associated with M_l and l and one can easily check that

$$\widetilde{\mathcal{E}}_l^{+} = \widetilde{M}_l + \tfrac{|\widetilde{l}|}{2} \quad \text{and} \quad \widetilde{\mathcal{E}}_l^{-} = \widetilde{M}_l - \tfrac{|\widetilde{l}|}{2}. \tag{2.13}$$

As a consequence, proving Theorem 2.5 is equivalent to proving that

$$\widehat{Q}_{L,\beta} \xrightarrow[L\to\infty]{d} \big(\xi_H, \xi_H^c\big), \tag{2.14}$$

where $\widehat{Q}_{L,\beta}$ is the law of $\sqrt{N_l}\big(\widetilde{M}_l(s), |\widetilde{l}(s)| - \gamma_\beta^*(s)\big)_{s\in[0,1]}$ with l sampled from $\widetilde{P}_{L,\beta}$. The convergence of $\widetilde{M}_l$ in (2.14) answers an open question raised in [4, Fig. 14 and Table II] where the process is referred to as the *center-of-mass walk*.

The extended phase ($\beta < \beta_c$)

When $\beta < \beta_c$ and under $P_{L,\beta}$, we have seen that a typical path l adopts an extended configuration, characterized by a number of horizontal steps of order L. We let $Q_{L,\beta}$ be the law of $\sqrt{N_l}\,\big(\widetilde{\mathcal{E}}_l^-(s), \widetilde{\mathcal{E}}_l^+(s)\big)_{s\in[0,1]}$ under $P_{L,\beta}$. We let also $(B_s)_{s\in[0,1]}$ be a standard Brownian motion.

Theorem 2.8. *For $\beta < \beta_c$, there exists a $\sigma_\beta > 0$ such that*

$$Q_{L,\beta} \xrightarrow[L\to\infty]{d} \sigma_\beta\big(B_s, B_s\big)_{s\in[0,1]}. \tag{2.15}$$

Remark 2.9. The constant σ_β takes value $\sqrt{\mathfrak{E}_\beta(y_1^2)/\mathfrak{E}_\beta(\nu_1)}$ where y_1 (resp. ν_1) corresponds to the vertical (resp. horizontal) displacement of the path on one of the pattern mentioned in Remark 2.3 (1). These objects are defined rigorously in Section 6 below.

2.3 Discussion and open problems

Giving a path characterization of the phase transition is an important issue for polymer models in Statistical Mechanics. From that point of view, identifying in each regime the limiting distribution of the whole path rescaled in time by its total length N and in space by some ad-hoc power of N is challenging and meaningful. This question was investigated in depth for $(1+1)$-dimensional wetting models, for instance in [13] and [8] when the pinning occurs at the x-axis (hard wall), in [32] when the pinning of the path occurs at a layer of finite width on top of the hard wall, or in [7] with some additional stiffness imposed on the trajectories of the path. Although the features of IPDSAW strongly differ from that of wetting models, we intend here to answer similar questions. Finally, the interest of our work is raised by the fact that the collapse transition undergone by the IPDSAW is fundamentally different from the localization transition displayed by wetting models and this can be explained in a few words. For the wetting model, the saturated phase for which the free energy is trivial (=0) corresponds to the polymer being fully delocalized off the interface which means that entropy completely takes over in the energy-entropy competition that rules such systems. For the IPDSAW in turn, the saturated phase is characterized by a domination of trajectories that are maximizing the energy. In other words, we could say that both models display a saturated phase which in the pinning case is associated with a maximization of the entropy, whereas it is associated with a maximization of the energy for the polymer collapse.

In our paper, we give a rather complete description of the scaling limits of IPDSAW in each regime. There are a few open questions left that are stated as open problems at the end of this section.

Critical regime. The most important result of the paper is concerned with the critical regime ($\beta = \beta_c$) for which we provide the limiting distribution of the horizontal extension rescaled by $L^{2/3}$ and the sharp asymptotic of the partition function. In Section 4, indeed, we use the random walk representation described in Section 3.1 and the fact that $\Gamma_{\beta_c} = 1$ to claim that the horizontal extension of the path has the law of the stopping time $\tau_L := \min\{N \geq 1 \colon G_N(V) \geq L - N\}$ when V is sampled from $\mathbf{P}_\beta(\cdot \,|\, V_{\tau_L} = 0, G_{\tau_L}(V) = L - \tau_L)$. Studying the scaling of τ_L requires to build up a renewal process based on the successive excursions made by V inside the lower half-plane or inside the upper half-plane. A sharp local limit theorem is therefore required for the area enclosed in between such an excursion and the x-axis and this is precisely the object of a recent paper [12]. Note that the fact that the horizontal extension fluctuates makes the scaling limits of the upper and lower envelopes of the path much harder to investigate at $\beta = \beta_c$. As soon as the limiting distribution of the rescaled horizontal extension is not constant, which is the case for the critical IPDSAW, one should indeed consider simultaneously the limiting distribution of $\widetilde{V}$, of $\widetilde{M}$, and of the rescaled horizontal extension. For this reason, we will state the investigation of the limiting distribution of the upper and lower envelopes of the critical IPDSAW as an open problem. Let us conclude by pointing out that the critical regime of a Laplacian (1+1)-dimensional pinning model that is investigated in [7] has somehow a similar flavor. More precisely, when the pinning term ε is switched off, the path can be viewed as the bridge of an integrated random walk and therefore scales like $N^{3/2}$. This scaling persists until ε reaches a critical value ε_c. At criticality, and once rescaled in time by N and in space $N^{\frac{3}{2}}/\log^{\frac{5}{2}}(N)$ the path is seen as the density of a signed measure μ_N on $[0,1]$. Then, μ_N that is build with a path sampled from the polymer measure, converges in distribution towards a random atomic measure on $[0,1]$. The atoms of the limiting distribution are generated by the longest excursions of the integrated walk, very much in the spirit of the limiting distribution in Theorem 2.2 (2) which can be written as a sum (see (4.46)) whose terms are associated with the longest excursions of the auxiliary random walk.

Collapsed regime. Due to the convergence of both envelopes towards deterministic Wulff shapes, the collapsed IPDSAW may be related to other models in Statistical Mechanics that are known to undergo convergence of interfaces towards deterministic Wulff shapes. This is the case for instance when considering a 2 dimensional bond percolation model in its percolation regime and conditioned on the existence of an open curve of the dual graph around the origin with a prescribed large area enclosed inside the curve (see [1]). A similar interface appears when considering the 2-dimensional Ising model in a big square box of size N at low temperature with no external field and $-$ boundary conditions and when conditioning the total magnetization to deviate from its average (i.e., $-m^*N^2$ with $m* > 0$) by a factor $a_N \sim N^{4/3+\delta}$ ($\delta > 0$). It has been proven in [16], [22], [23] and [24] that such a deviation is typically due to a unique large droplet of $+$, whose boundary converges to a deterministic Wulff shape once rescaled by $\sqrt{a_N}$.

However, the closest relatives to the collapsed IPDSAW are probably the 1-dimensional SOS model with a prescribed large area below the interface (see [14]) and the 2-dimensional Ising interface separating the $+$ and $-$ phases in a vertically infinite layer of finite width (again with a large area underneath the interface, see [15]). For both models and in size $N \in \mathbb{N}$, the law of the interface can be related to the law of an underlying random walk V conditioned on describing an abnormally large *algebraic* area (qN^2 with $q > 0$). As a consequence, once rescaled in time and space by N the interface converges in probability towards a Wulff shape, whose characteristics depends on q and on the random walk distribution. The fluctuations of the interface around this deterministic shape are of order $\sqrt{N}$ and their limiting distribution is identified in [14, theorem 2.1] for SOS model and in [15, theorem 3.2] for Ising interface at sufficiently low temperature. The proofs in [14] and [15] use an ad-hoc tilting of the random walk law (described in Section 5.1), so that the large area becomes typical under the tilted law. In this framework, a local limit theorem can be derived for any finite dimensional distribution of V under the tilted law.

In the present paper, our system also enjoys a random walk representation (see Section 3.1) and we will use the "large area" tilting of the random walk law as well to prove Theorem 3.1. However, our model displays three particular features that prevent us from applying the results of [14] straightforwardly. First, the conditioning on the auxiliary random walk V is, in our case, related to the *geometric* area below the path rather than to the *algebraic* area (see Remark 2.6). Second, the horizontal extension of an IPDSAW path fluctuates, which is not the case for SOS model. Thus, the ratio q of the area below the path divided by the square of its horizontal extension fluctuates as well, which forces us to display some uniformity in q for every local limit theorem we state in Section 5. Third and last, the fact that an IPDSAW path is characterized by two envelopes makes it compulsory to study simultaneously the fluctuations of V around the Wulff shape and the fluctuations of M around the x-axis (recall (2.12–2.14)). We recall that the increments of M are obtained by switching the sign of every second increment of V. As a consequence, we need to adapt, in Section 5.1, the proofs of the finite dimensional convergence and of the tightness displayed in [14].

We need to mention that in the collapsed phase, our asymptotics, (3) of Theorem 2.1, are less precise that the asymptotics of [21, Theorem 1.1] where they give exact asymptotics, with a square root prefactor for the partition function of their self-avoiding polygons model. A reason we could not adapt easily their results/methods is that, among other differences, in our model we penalize the horizontal steps by a factor Γ_β that differs from the penalization we assign to the vertical steps (increments of a RW of law $\mathbf{P}_\beta$) whereas in their model each step is penalized by the same factor $e^{-\beta}$.

Extended regime. The extended regime is somehow easier to deal with. One can indeed decompose the trajectory into simple patterns, that do not interact with each other and are typically of finite length, that is, the pieces of path in between two consecutive vertical stretches of length 0. We will briefly show in Section 6 that these patterns can be seen as independent building blocks of the path and can be associated with a positive recurrent renewal.

Computer Simulations

As explained in the Appendix 7, the representation formula (1.7) provides an exact simulation algorithm for the law of a path under the polymer measure $P_{\beta,L}$. However, this algorithm is very efficient only for $\beta = \beta_c$, and loses all efficiency when β is not close to β_c.

Open problems

- Find the scaling limit of the envelopes of the path in the critical regime.
- Establish the fluctuations of the envelopes around the Wulff shapes (Theorem 2.5) for the true polymer measure $P_{L,\beta}$ rather that for the mixture $\widetilde{P}_{L,\beta}$.
- Establish a Central Limit Theorem and a Large Deviation Principle for the horizontal extensions in the collapsed and extended regimes.
- Devise a dynamic scheme of convergence of measures on paths such that the equilibrium measure is the polymer measure, and with a sharp control on the mixing time to equilibrium similar to the one devised for S.O.S in [5].

3 Preparation

In this section, we recall the proof of the probabilistic representation of the partition function (recall 1.7). This proof was already displayed in [29] but since it constitutes the starting point of our analysis it is worth reproducing it here briefly. Moreover, we obtain as a by product the auxiliary random walk V of law $\mathbf{P}_\beta$, which, under an appropriate conditioning, can be used to derive some path properties under the polymer measure.

3.1 Probabilistic representation of the partition function

Let us recall that $V := (V_n)_{n\in\mathbb{N}}$ is a random walk of law $\mathbf{P}_\beta$ satisfying $V_0 = 0$, $V_n = \sum_{i=1}^n U_i$ for $n \in \mathbb{N}$ and $(U_i)_{i\in\mathbb{N}}$ is an i.i.d sequence of integer random variables whose law was defined in (1.6). For $L \in \mathbb{N}$ and $N \in \{1,\dots,L\}$ we recall that

$$\mathcal{V}_{N,L-N} := \{V : G_N(V) = L-N, V_{N+1} = 0\} \quad \text{with} \quad G_N(V) = \textstyle\sum_{i=0}^N |V_i|,$$

and (see Fig. 4) we denote by T_N the one-to-one correspondence that maps $\mathcal{V}_{N,L-N}$ onto $\mathcal{L}_{N,L}$ as

$$T_N(V)_i = (-1)^{i-1} V_i \quad \text{for all} \quad i \in \{1,\dots N\}. \tag{3.1}$$

Coming back to the proof of (1.7) we recall (1.1–1.5) and we note that $\forall x, y \in \mathbb{Z}$ one can write $x \,\widetilde{\wedge}\, y = \frac{1}{2}(|x| + |y| - |x+y|)$. Hence, for $\beta > 0$ and $L \in \mathbb{N}$, the partition function in (1.5) becomes

$$\begin{aligned} Z_{L,\beta} &= \sum_{N=1}^{L} \sum_{\substack{l\in\mathcal{L}_{N,L}\\ l_0=l_{N+1}=0}} \exp\Big(\beta \sum_{n=1}^N |l_n| - \frac{\beta}{2} \sum_{n=0}^N |l_n + l_{n+1}|\Big) \\ &= c_\beta\, e^{\beta L} \sum_{N=1}^{L} \big(\tfrac{c_\beta}{e^\beta}\big)^N \sum_{\substack{l\in\mathcal{L}_{N,L}\\ l_0=l_{N+1}=0}} \prod_{n=0}^{N} \frac{\exp\Big(-\frac{\beta}{2}|l_n + l_{n+1}|\Big)}{c_\beta}. \end{aligned} \tag{3.2}$$

Then, since for $l \in \mathcal{L}_{N,L}$ the increments $(U_i)_{i=1}^{N+1}$ of $V = (T_N)^{-1}(l)$ in (3.1) necessarily satisfy $U_i := (-1)^{i-1}(l_{i-1} + l_i)$, one can rewrite (3.2) as

$$Z_{L,\beta} = c_\beta e^{\beta L} \sum_{N=1}^{L} \big(\tfrac{c_\beta}{e^\beta}\big)^N \sum_{V\in\mathcal{V}_{N,L-N}} \mathbf{P}_\beta(V), \tag{3.3}$$

which immediately implies (1.7). A useful consequence of formula (3.3) is that, once conditioned on taking a given number of horizontal steps N, the polymer measure is exactly the image measure by the T_N-transformation of the geometric random walk V conditioned to return to the origin after $N+1$ steps and to make a geometric area $L-N$, i.e.,

$$P_{L,\beta}\big(l \in \cdot \mid N_l = N\big) = \mathbf{P}_\beta\big(T_N(V) \in \cdot \mid V_{N+1} = 0, G_N = L-N\big). \tag{3.4}$$

4 Scaling Limits in the critical phase

In this section we will prove the items (2) of Theorems 2.1 and 2.2 which correspond to the critical case ($\beta = \beta_c$). To simplify notations, we shall write β instead of β_c until the end of this proof. In Section 4.1 below, we first exhibit a renewal structure for the underlying geometric random walk, based on "excursions". Then we state a local limit theorem for the area of such an excursion. This Theorem has been proven recently in [12]. With these tools in hand we will be able to prove Theorem 2.1 (2) in Section 4.2 and Theorem 2.2 (2) in Section 4.3. Finally, in Section 4.4, we identify the limiting law of the rescaled horizontal extension obtained in Section 4.3 with that of the Brownian stopping time g_1 under a proper conditioning.

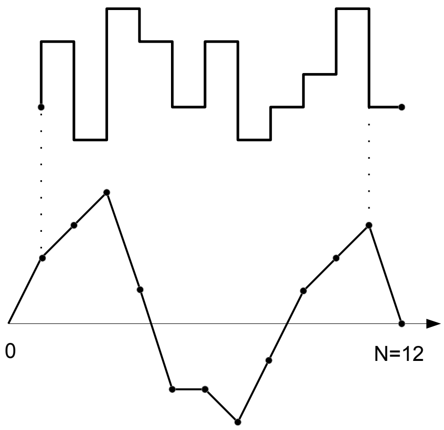

Figure 4: An example of a trajectory $l = (l_i)_{i=1}^{11}$ with $l_1 = 2, l_2 = -3, l_4 = 4 \ldots$ is drawn on the upper picture. The auxiliary random walk V associated with l, i.e., $(V_i)_{i=0}^{12} = (T_{11})^{-1}(l)$ is drawn on the lower picture with $V_1 = 2, V_2 = 3, V_4 = 4 \ldots$

4.1 Preparations

The renewal structure

We introduce a sequence of stopping times $(\tau_k)_{k\in\mathbb{N}}$ which are similar to ladder times. To be more specific we set $\tau_0 = 0$ and

$$\tau_{k+1} = \inf\left\{i > \tau_k : V_{i-1} \neq 0 \text{ and } V_{i-1}V_i \leq 0\right\} . \tag{4.1}$$

To these we associate, for $j \in \mathbb{N}$, the length of the jth inter-arrival of $(\tau_k)_{k\in\mathbb{N}}$, i.e.,

$$\mathfrak{N}_j = \tau_j - \tau_{j-1} \quad (j \geq 1)\,, \tag{4.2}$$

and the associated geometric area

$$\mathfrak{A}_j = \left|V_{\tau_{j-1}}\right| + \cdots + \left|V_{\tau_j - 1}\right| \quad (j \geq 1)\,. \tag{4.3}$$

We let $\tau = \tau_1 = \mathfrak{N}_1$. Let us observe that $\mathfrak{A}_1 = G_{\tau-1}(V)$. For simplicity, we will drop the V dependency of G in what follows.

Now, we state the main result of this section, and the remaining part of this section is devoted to its proof.

Proposition 4.1. *The random variables $(\mathfrak{A}_i, \mathfrak{N}_i)_{i\geq 1}$ are independent and the sequence $(\mathfrak{A}_i, \mathfrak{N}_i)_{i\geq 2}$ is IID.*

We shall first need to study the distribution of V_τ. Let T be a random variable with geometric distribution with parameter $p_\beta = 1 - e^{-\beta/2}$ that is

$$\mathbb{P}\left(T = k\right) = e^{-\frac{\beta}{2}k}(1 - e^{-\beta/2}) \qquad (k \in \mathbb{N} \cup \{0\})\,,$$

and let μ_β be the law of the associated symmetric random variable, that is μ_β is the distribution of εT with ε independent from T and $\mathbb{P}\left(\varepsilon = \pm 1\right) = \frac{1}{2}$:

$$\mu_\beta(k) = \mathbb{P}\left(\varepsilon T = k\right) = \frac{1 - e^{-\beta/2}}{2} e^{-\frac{\beta}{2}|k|} 1_{(k\neq 0)} + (1 - e^{-\beta/2}) 1_{(k=0)}. \tag{4.4}$$

Finally we let $\mathbf{P}_{\beta,x}$ be the law of the random walk starting from $V_0 = x \in \mathbb{Z}$ and $\mathbf{P}_{\beta,\mu_\beta}$ be the law of the random walk when V_0 has distribution μ_β.

Lemma 4.2. *Under $\mathbf{P}_{\beta,x}$ with $x \in \mathbb{Z}$ or under $\mathbf{P}_{\beta,\mu_\beta}$, the random variable V_τ is independent from the couple $(G_{\tau-1}, \tau)$. Moreover*

- $V_\tau =_{law} -T$ *under* $\mathbf{P}_{\beta,x}$ *with* $x > 0$,
- $V_\tau =_{law} T$ *under* $\mathbf{P}_{\beta,x}$ *with* $x < 0$,
- $V_\tau =_{law} \mu_\beta$ *under* $\mathbf{P}_{\beta}$,
- $V_\tau =_{law} \mu_\beta$ *under* $\mathbf{P}_{\beta,\mu_\beta}$.

Proof. Let $x > 0, y \geq 0$ and a, n be integers. Under $\mathbf{P}_{\beta,x}$ we compute the probability of $T_{n,a,y} := \{G_{\tau-1} = a, \tau = n, V_\tau = -y\}$ by disintegrating it with respect to the value $z > 0$ taken by V_{n-1}, i.e.,

$$\begin{aligned}
&\mathbf{P}_{\beta,x}(G_{\tau-1} = a, \tau = n, V_\tau = -y) \\
&\qquad = \sum_{z>0} \mathbf{P}_{\beta,x}\left(V_1 + \cdots + V_{n-1} = a; V_i > 0, 1 \leq i \leq n-2; V_{n-1} = z\right) \frac{e^{-\frac{\beta}{2}(z+y)}}{c_\beta} \\
&\qquad = \gamma_\beta \mathbf{P}_{\beta,x}\left(G_{\tau-1} = a, \tau = n\right) e^{-\frac{\beta}{2}y}\,, \qquad (4.5)
\end{aligned}$$

where γ_β can be seen as a normalizing constant for a distribution on the non negative integers, we obtain that $\gamma_\beta = p_\beta$ and V_τ is independent of $(G_{\tau-1}, \tau)$ and distributed as $-T$. For $x < 0$ the proof is exactly the same. For $x = 0$, we take into account the possibility that the walk sticks to zero for a while. Thus, for $y > 0$, we partition the event $\{G_{\tau-1} = a, \tau = n, V_\tau = -y\}$ depending on the value $z > 0$ taken by V_{n-1} and on the number of steps k during which the random walks sticks to 0, i.e.,

$$\begin{aligned}
&\mathbf{P}_{\beta}\left(G_{\tau-1} = a, \tau = n, V_\tau = -y\right) \\
&= \sum_{z>0, 0\leq k\leq n-2} \mathbf{P}_{\beta,x}\left(A_{n-1} = a; V_i = 0, 0 \leq i \leq k; V_i > 0, k < i \leq n-2; V_{n-1} = z\right) \frac{e^{-\frac{\beta}{2}(z+y)}}{c_\beta} \\
&:= \kappa e^{-\frac{\beta}{2}y}, \qquad (4.6)
\end{aligned}$$

where κ is implicitly defined by (4.6) and its dependence in a and n is omitted for simplicity. We obtain by symmetry $\mathbf{P}_\beta\left(G_{\tau-1} = a, \tau = n, V_\tau = y\right) = \kappa e^{-\frac{\beta}{2}y}$. It is only for $x = y = 0$ that we need to take into account positive and negative excursions, and we obtain

$$\mathbf{P}_\beta\left(G_{\tau-1} = a, \tau = n, V_\tau = 0\right) = 2\kappa\,.$$

Summing all these probabilities yields

$$\kappa = \frac{1 - e^{-\beta/2}}{2} \mathbf{P}_\beta\left(G_{\tau-1} = a, \tau = n\right),$$

and we can conclude that the random variable V_τ is independent from the couple $(G_{\tau-1}, \tau)$ with distribution μ_β.

The final computation showing that μ_β is an “invariant measure”, is straightforward:

$$\begin{aligned}
\mathbf{P}_{\beta,\mu_\beta}\left(V_\tau = y\right) &= \sum_x \mu_\beta(x) \mathbf{P}_{\beta,x}\left(V_\tau = y\right) && (4.7)\\
&= \sum_x \mu_\beta(x) (\mathbb{P}\left(-T = y\right) 1_{(x>0)} + \mathbb{P}\left(T = y\right) 1_{(x<0)} + \mu_\beta(y) 1_{(x=0)}) && (4.8)\\
&= \ldots = \mu_\beta(y). && (4.9)
\end{aligned}$$

□

Proof of Proposition 4.1. The proof is based on the preceding Lemma, and uses induction in conjunction with the Markov property. Our induction assumption is thus that the sequence $(\mathfrak{A}_i, \mathfrak{N}_i)_{1\le i\le k}$ is independent, that the subsequence $(\mathfrak{A}_i, \mathfrak{N}_i)_{2\le i\le k}$ is IID, and that the random variable V_{τ_k} is independent of $(\mathfrak{A}_i, \mathfrak{N}_i)_{1\le i\le k}$, with distribution μ_β. Then,

$$\begin{aligned}\mathbf{P}_\beta\left((\mathfrak{A}_i, \mathfrak{N}_i) = (a_i, n_i), 1\le i\le k+1; V_{\tau_{k+1}} = y\right) = \\ \mathbf{E}_\beta\left[1_{\{(\mathfrak{A}_i,\mathfrak{N}_i)=(a_i,n_i),1\le i\le k\}}\mathbf{P}_{V_{\tau_k}}\left((\mathfrak{A}_1, \mathfrak{N}_1) = (a_{k+1}, n_{k+1}); V_\tau = y\right)\right] \\ = \mu_\beta(y)\mathbf{P}_{\beta,\mu_\beta}\left(A_{\tau-1} = a_{k+1}, \tau = n_{k+1}\right) \mathbf{P}_\beta\left((\mathfrak{A}_i, \mathfrak{N}_i) = (a_i, n_i), 1\le i\le k\right) ,\end{aligned}$$

and this concludes the induction step. □

Remark 4.3. The renewal structure used in the present paper is somehow reminiscent of another renewal structure used in [33], where the author focuses on the probability that different types of integrated random walks remain positive up to an arbitrary large time n. The random walk of law $\mathbf{P}_\beta$ that we consider here falls into the scope of this work. In [33], the random walk paths are decomposed into cycles containing each a positive and a negative excursion. Although the path decomposition applied in the present work is different since for instance our excursions are not necessarily of alternating signs, and although we are more focused here on establishing local limit theorems, both works rely on a particular feature associated with the type of random walks considered, namely, the statement of our Lemma 4.2 or in words, as stated in [33] “the overshoot over any fixed level is independent of the moment when it occurs and also of the walk up to this moment”.

Local limit theorem for the “excursion” area

Let us state first the theorem. Here f_{ex} stands for the density of the area of the standard Brownian excursion (see e.g. [25]).

Theorem 4.4 (Theorem 1 of [12]). *With the constant $C_\beta := (\mathbf{E}_\beta(V_1^2))^{-1/2}$ and with $w(x) = C_\beta\, f_{ex}(C_\beta\, x)$, we have*

$$\lim_{n\to+\infty}\sup_{a\in\mathbb{Z}}\left|n^{3/2}\mathbf{P}_\beta\left(G_n = a \mid \tau = n\right) - w(a/n^{3/2})\right| = 0\,.$$

Remark 4.5. From the monograph [25] we extract the asymptotics (formulas (93) and (96))

$$f_{ex}(x) = e^{-\frac{C_1}{x^2}}(C_2x^{-5} + o(x^{-5}) \quad (x\to 0) \tag{4.10}$$

$$f_{ex}(x) = C_3x^2e^{-6x^2}(1 + o(1)) \quad (x\to+\infty). \tag{4.11}$$

A straightforward application of the dominated convergence theorem entails that if $b_n\to\infty$ then for all $\eta > 0$,

$$\frac{1}{n^{2/3}}\sum_{\eta n^{2/3}\le k\le b_n n^{2/3}}\left(\frac{k}{n^{2/3}}\right)^{-3} w\left(\left(\frac{k}{n^{2/3}}\right)^{-3/2}\right) \to \int_\eta^{+\infty} t^{-3}w(t^{-3/2})\,dt. \tag{4.12}$$

It is easy to check from [12] that this theorem still holds when started from the “invariant measure” μ_β. More precisely, $R_n\to 0$ with

$$R_n := \sup_{a\in\mathbb{Z}}\left|n^{3/2}\mathbf{P}_{\beta,\mu_\beta}\left(G_n = a \mid \tau = n\right) - w(a/n^{3/2})\right|. \tag{4.13}$$

4.2 Proof of Theorem (2.1) (2)

Using the random walk representation (3.3), we obtain, since $\Gamma_\beta = 1$, that the excess partition function is

$$\tfrac{1}{c_\beta}\widetilde{Z}_{L,\beta} = \sum_{N=1}^{L} \mathbf{P}_\beta\left(G_N = L-N, V_{N+1}=0\right). \tag{4.14}$$

Then, we partition the event $\{G_N = L-N, V_{N+1}=0\}$ with respect to $r := \inf\{i \geq 0\colon\ V_{N-i} \neq 0\}$ the time length during which the random walk sticks to the origin before its N-th step, i.e.,

$$\begin{aligned}\tfrac{1}{c_\beta}\widetilde{Z}_{L,\beta} &= \sum_{r=0}^{L-1}\sum_{N=1}^{L-r} \mathbf{P}_\beta\left(G_N = L-N-r, V_N \neq 0, V_{N+1}=0\right)\mathbf{P}_\beta\left(V_1=\cdots=V_r=0\right) \quad (4.15)\\ &\qquad + \mathbf{P}_\beta\left(V_1=\cdots=V_L=0\right)\\ &= \left(\tfrac{1}{c_\beta}\right)^L + \sum_{r=0}^{L-1}\left(\tfrac{1}{c_\beta}\right)^r \sum_{N=1}^{L-r}\mathbf{P}_\beta\left(G_N = L-N-r, V_N\neq 0, V_{N+1}=0\right).\end{aligned}$$

Then we use the fact that, for all $x \in \mathbb{N}$, we have $\mathbf{P}_\beta\left(U_1 = x\right)/\mathbf{P}_\beta\left(U_1 \geq x\right) = 1 - e^{-\beta/2}$, and obtain

$$\begin{aligned}\tfrac{1}{c_\beta}\widetilde{Z}_{L,\beta} &= \left(\tfrac{1}{c_\beta}\right)^L + (1-e^{-\beta/2})\sum_{r=0}^{L-1}\left(\tfrac{1}{c_\beta}\right)^r \sum_{N=1}^{L-r}\mathbf{P}_\beta\left(G_N = L-N-r, V_N\neq 0, V_N V_{N+1}\leq 0\right)\\ &= \left(\tfrac{1}{c_\beta}\right)^L + (1-e^{-\beta/2})\sum_{r=0}^{L-1}\left(\tfrac{1}{c_\beta}\right)^r \mathbf{P}_\beta\left(L-r+1 \in \mathfrak{X}\right), \quad (4.16)\end{aligned}$$

where

$$\mathfrak{X} = \Big\{\sum_{k\leq n} \mathfrak{A}_k + \mathfrak{N}_k; n \geq 1\Big\} \tag{4.17}$$

is the renewal set associated to the sequence of random variables $X_k := \mathfrak{A}_k + \mathfrak{N}_k$ (recall (4.1–4.3).

It is clear that we are going to obtain the same asymptotics for $\widetilde{Z}_{L,\beta}$ if we substitute $\mathbf{P}_{\beta,\mu_\beta}$ to $\mathbf{P}_\beta$ in the r.h.s. of (4.16), that is if we consider a true renewal process with the random variable X_1 having the same distribution as the X_i for $i \geq 2$. Thus, the proof of Theorem (2.1) (2) will be a consequence of the tail estimate of X under $\mathbf{P}_{\beta,\mu_\beta}$ in the next lemma.

Lemma 4.6. *For $\beta > 0$, there exists a $c_{1,\beta} > 0$ such that*

$$\mathbf{P}_{\beta,\mu_\beta}(X_1 = n) = \frac{c_{1,\beta}}{n^{4/3}}(1+o(1)), \tag{4.18}$$

and $c_{1,\beta} = (1+e^{\beta/2})\sqrt{\frac{\mathbf{E}_\beta[V_1^2]}{2\pi}}\int_0^{+\infty} x^{-3} w(x^{-\frac{3}{2}})\,dx$.

By applying [17, Theorem B] (see also Theorem A.7 of [19]) we deduce from (4.18) that

$$\mathbf{P}_{\beta,\mu_\beta}(L \in \mathfrak{X}) = \frac{\sin(\pi/3)}{3\pi c_{1,\beta} L^{2/3}}(1+o(1)). \tag{4.19}$$

Then, it suffices to recall (4.16-4.17) to complete the proof.

Proof of Lemma 4.6. We recall that $\tau_1 = \mathfrak{N}_1$ and we drop the index 1 for simplicity. First, we use that for all $j, k \in \mathbb{N}$

$$\frac{\mathbf{P}_{\beta,\mu_\beta}\left(V_{j+1}-V_j \geq k\right)}{\mathbf{P}_{\beta,\mu_\beta}\left(V_{j+1}-V_j = k\right)} = \frac{1}{1-e^{-\frac{\beta}{2}}}$$

to write

$$\mathbf{P}_{\beta,\mu_\beta}\left(X_1=n\right)=\mathbf{P}_{\beta,\mu_\beta}\left(A_{\tau-1}+\tau=n\right)=\mathbf{P}_{\beta,\mu_\beta}\left(A_{\tau-1}+\tau=n,V_\tau=0\right)\tfrac{1}{1-e^{-\frac{\beta}{2}}} \tag{4.20}$$

and then, for $\eta\in(0,1)$, we split the probability of the r.h.s. in (4.20) into two terms:

$$\begin{aligned}
&\mathbf{P}_{\beta,\mu_\beta}\left(A_{\tau-1}+\tau=n,V_\tau=0\right)\\
&=\sum_{k=1}^{\eta n^{2/3}}\mathbf{P}_{\beta,\mu_\beta}\left(A_{k-1}=n-k,\tau=k,V_k=0\right)+\sum_{k=\eta\, n^{2/3}}^{n}\mathbf{P}_{\beta,\mu_\beta}\left(A_{k-1}=n-k,\tau=k,V_k=0\right)\\
&=: u_n+v_n.
\end{aligned} \tag{4.21}$$

From this equality, the proof will be divided into two steps. The first step consists in controlling u_n and the second step v_n.

Step 1

Our aim is to show that for all $\varepsilon>0$ there exists an $\eta>0$ such that

$$\limsup_{n\to\infty} n^{4/3}u_n\leq\varepsilon. \tag{4.22}$$

Proof. In this step, we will need an improved version of the local limit Theorem established in [6, Proposition 2.3] for V_n and A_n simultaneously. The proof is given in the Appendix 7.3.

Proposition 4.7.

$$\sup_{n\in\mathbb{N}}\sup_{k,a\in\mathbb{Z}} n^3\left|\mathbf{P}_\beta(V_n=k,A_n(V)=a)-\tfrac{1}{\sigma_\beta^2\,n^2}\,g\Big(\tfrac{k}{\sigma_\beta\sqrt{n}},\tfrac{a}{\sigma_\beta n^{3/2}}\Big)\right|<\infty, \tag{4.23}$$

with $g(y,z)=\frac{6}{\pi}e^{-2y^2-6z^2+6yz}$ *for* $(y,z)\in\mathbb{R}^2$.

We resume the proof of (4.22) by bounding from above the probability that there exists a piece of the $V=(V_i)_{i=0}^n$ trajectory of length smaller than $\frac{n^{2/3}}{\log n}$ with an algebraic area (seen from its starting point) that is larger than $\frac{n}{2}$ and/or that one of the increments of V is larger than $(\log n)^2$. Thus, we set $\mathcal{B}_n:=\mathcal{C}_n\cup\mathcal{D}_n$ with

$$\mathcal{C}_n:=\bigcup_{i\in\{0,\dots,n-1\}}\{V_{i+1}-V_i\geq(\log n)^2\}\ \text{and}\ \mathcal{D}_n:=\bigcup_{(j_1,j_2)\in J_n}\{A_{j_1,j_2}-V_{j_1}(j_2-j_1)\geq\tfrac{n}{2},V_{j_1}\geq 0\}$$

where $J_n=\big\{(j_1,j_2)\in\{0,\dots,n\}^2\colon 0\leq j_2-j_1\leq\frac{n^{2/3}}{\log n}\big\}$ and $A_{s,t}=\sum_{i=s}^{t-1}V_i$. Then, for each $(j_1,j_2)\in J_n$ we apply Markov property at j_1 and we get

$$\mathbf{P}_{\beta,\mu_\beta}(\mathcal{D}_n)\leq\sum_{(j_1,j_2)\in J_n}\mathbf{P}_\beta(A_{j_2-j_1}\geq\tfrac{n}{2}). \tag{4.24}$$

Since under $\mathbf{P}_{\beta,\mu_\beta}$, the random variable V_1 has small exponential moments, there exists a constant $C>0$ such that

$$\mathbf{P}_{\beta,\mu_\beta}\left(\sup_{1\leq k\leq n}|V_k|\geq xn\right)\leq 2e^{-Cnx\wedge x^2}\qquad(x\geq 0,n\in\mathbb{N})\,. \tag{4.25}$$

We note that $A_{j_2-j_1}\geq\frac{n}{2}$ implies $\max\{|V_i|,i=1,\dots,j_2-j_1\}\geq\frac{n}{2(j_2-j_1)}$ so that finally we can use (4.25) to prove that there exists $C''>0$ such that

$$\begin{aligned}
\sup_{(j_1,j_2)\in J_n}\mathbf{P}_\beta(A_{j_2-j_1}\geq\tfrac{n}{2})&\leq\mathbf{P}_\beta\bigg(\sup_{1\leq j\leq n^{2/3}/\log(n)}|V_j|\geq\tfrac{n^{1/3}}{2}\log(n)\bigg)\\
&\leq 2e^{-C''\log(n)^3},
\end{aligned} \tag{4.26}$$

which (recall (4.24)) suffices to claim that $\mathbf{P}_{\beta,\mu_\beta}(\mathcal{D}_n) = o(1/n^{4/3})$. Moreover, $\mathbf{P}_{\beta,\mu_\beta}(V_1 \geq (\log n)^2) \leq ce^{-\frac{\beta}{2}\log(n)^2}$ suffices to conclude that $\mathbf{P}_{\beta,\mu_\beta}(\mathcal{C}_n) = o(1/n^{4/3})$ which, in turn, implies that $\mathbf{P}_{\beta,\mu_\beta}(\mathcal{B}_n) = o(1/n^{4/3})$.

At this stage, for $k \leq \eta n^{2/3}$, we can partition the set $\{A_{k-1} = n-k, \tau = k, V_k = 0\}$ depending on the indices at which a trajectory passes above $\sqrt{k}$ for the first and the last time. Thus, we set $\xi_{\sqrt{k}} = \inf\{i \geq 0\colon\ V_i \geq \sqrt{k}\}$ and $\widehat{\xi}_{\sqrt{k}} = \max\{i \leq k\colon\ V_i \geq \sqrt{k}\}$. We also consider the positions of V at $\xi_{\sqrt{k}}$ and $\widehat{\xi}_{\sqrt{k}}$ and the algebraic areas below V in-between 0 and $\xi_{\sqrt{k}}$, $\xi_{\sqrt{k}}$ and $\widehat{\xi}_{\sqrt{k}}$ as well as $\widehat{\xi}_{\sqrt{k}}$ and k. Thus, we set $\bar{t} = (t_1, t_2)$, $\bar{x} = (x_1, x_2)$, $\bar{a} = (a_1, a_2)$ and we write

$$\{A_{k-1} = n-k, \tau = k, V_k = 0\} = \cup_{(\bar{t},\bar{x},\bar{a})\in G_{k,n}} Y_{n,k}(\bar{t},\bar{x},\bar{a}), \tag{4.27}$$

with

$$\begin{aligned} Y_{n,k}(\bar{t},\bar{x},\bar{a}) := \Big\{&\tau = k, \xi_{\sqrt{k}} = t_1,\, \widehat{\xi}_{\sqrt{k}} = k - t_2, \\ &V_{t_1} = x_1, V_{k-t_2} = x_2, V_k = 0, \\ &A_{t_1} = a_1, A_{t_1,k-t_2} = n-k-a_1-a_2, A_{k-t_2,k} = a_2\Big\}, \end{aligned} \tag{4.28}$$

and with

$$\begin{aligned} G_{k,n} = \Big\{&(\bar{t},\bar{x},\bar{a}) \in \mathbb{N}^6\colon\ 0 \leq t_1 \leq k - t_2 \leq k-1, \\ &0 \leq a_1 \leq k^{3/2},\ 0 \leq a_2 \leq k^{3/2}, \\ &x_1 \geq \sqrt{k},\ x_2 \geq \sqrt{k}\Big\}. \end{aligned} \tag{4.29}$$

Then we set

$$\widetilde{G}_{k,n} = \big\{(\bar{t},\bar{x},\bar{a}) \in G_{k,n}\colon\ k - t_1 - t_2 \leq \tfrac{n^{2/3}}{\log n} \quad \text{or} \quad x_1 - \sqrt{k} > \log(n)^2\big\}, \tag{4.30}$$

and we note that if $(\bar{t},\bar{x},\bar{a}) \in \cup_{k=1}^{\eta n^{2/3}} \widetilde{G}_{k,n}$, then either $x_1 - \sqrt{k} \geq (\log n)^2$, which implies in particular $Y_{n,k}(\bar{t},\bar{x},\bar{a}) \subset \mathcal{C}_n$, or $x_1 \leq \sqrt{k} + (\log n)^2$ and then

$$\begin{aligned} A_{t_1,k-t_2} - V_{t_1}(k-t_1-t_2) &= n-k-a_1-a_2-x_1(k-t_1-t_2) \\ &\geq n-k-3k^{3/2}-(\log n)^2 k \\ &\geq n-\eta n^{2/3}-3\eta^{3/2}n-\log(n)^2\eta n^{2/3} \geq \frac{n}{2}, \end{aligned} \tag{4.31}$$

provided η is chosen small enough. Thus, $Y_{n,k}(\bar{t},\bar{x},\bar{a}) \subset \mathcal{D}_n$ so that

$$\cup_{k=1}^{\eta n^{2/3}} \cup_{(\bar{t},\bar{x},\bar{a})\in\widetilde{G}_{k,n}} Y_{n,k}(\bar{t},\bar{x},\bar{a}) \subset \mathcal{B}_n.$$

Clearly, for $k \leq n^{2/3}/\log(n)$ we have $G_{k,n} = \widetilde{G}_{k,n}$ so that we should simply focus on bounding from above

$$\mathbf{P}_{\beta,\mu_\beta}\Big(\cup_{k=\frac{n^{2/3}}{\log n}}^{\eta n^{2/3}} \cup_{(\bar{t},\bar{x},\bar{a})\in G_{k,n}\setminus\widetilde{G}_{k,n}} Y_{n,k}(\bar{t},\bar{x},\bar{a})\Big).$$

Pick $n^{2/3}/\log n \leq k \leq \eta n^{2/3}$ and $(\bar{t},\bar{x},\bar{a}) \in G_{k,n} \setminus \widetilde{G}_{k,n}$. By applying Markov property at t_1 and $k-t_2$ and by applying time reversibility between $k-t_2$ and k, we can write

$$\mathbf{P}_{\beta,\mu_\beta}\left(Y_{n,k}(\bar{t},\bar{x},\bar{a})\right) = S_1 S_2 S_3 \tag{4.32}$$

with

$$
\begin{aligned}
&S_1 = \mathbf{P}_{\beta,\mu_\beta}(\tau > t_1, V_{t_1} = x_1, \xi_{\sqrt{k}} = t_1, A_{t_1} = a_1)\\
&S_2 = \mathbf{P}_{\beta,x_1}(\tau > k - t_1 - t_2, V_{k-t_1-t_2} = x_2 - x_1, A_{k-t_1-t_2} = n - k - a_1 - a_2)\\
&S_3 = \mathbf{P}_{\beta}(\tau > t_2, V_{t_2} = x_2, \xi_{\sqrt{k}} = t_2, A_{t_2} = a_2).
\end{aligned}
\tag{4.33}
$$

Since we are looking for an upper bound of the r.h.s. in (4.32), we can remove the restriction $\{\tau > k - t_1 - t_2\}$ in S_2 and write

$$
S_2 \le \mathbf{P}_{\beta}(V_{k-t_1-t_2} = x_2 - x_1, A_{k-t_1-t_2} = n - k - a_1 - a_2 - x_1(k - t_1 - t_2)). \tag{4.34}
$$

Therefore, it remains to bound

$$
\sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \sum_{(\bar{t},\bar{x},\bar{a})\in G_{k,n}\setminus \widetilde{G}_{k,n}} S_1 S_2 S_3
$$

and we recall (4.34) and Proposition 4.7 that yield $S_2 \le \widetilde{S}_2 + \widehat{S}_2$ with $\widetilde{S}_2 = \frac{C_1}{(k-t_1-t_2)^3}$ and

$$
\widehat{S}_2 = \tfrac{C_2}{(k-t_1-t_2)^2}\, g\big(\tfrac{x_2-x_1}{\sigma_\beta\sqrt{k-t_1-t_2}}, \tfrac{n-k-a_1-a_2-x_1(k-t_1-t_2)}{\sigma_\beta (k-t_1-t_2)^{3/2}}\big), \tag{4.35}
$$

with $g(y,z) = \frac{6}{\pi} e^{-2y^2-6z^2+6yz} \le \frac{6}{\pi} e^{-\frac{3}{2}z^2}$. We recall (4.31) and we write

$$
\widehat{S}_2 \le \tfrac{C_2}{(k-t_1-t_2)^2} e^{-\frac{3}{2}\frac{n^2}{4\sigma_\beta^2 (k-t_1-t_2)^3}}, \tag{4.36}
$$

and then for all $(\bar{t},\bar{x},\bar{a}) \in G_{k,n} \setminus \widetilde{G}_{k,n}$ we have $\widetilde{S}_2 \le C_1 \log(n)^3/n^2$ and at the same time

$$
\sum_{(\bar{t},\bar{x},\bar{a})\in G_{k,n}} S_1 S_3 = \sum_{0\le t_1\le k-1} \mathbf{P}_{\beta,\mu_\beta}(\tau > t_1, \xi_{\sqrt{k}} = t_1, A_{t_1} \le k^{3/2}) \sum_{1\le t_2\le k-t_1} \mathbf{P}_{\beta}(\tau > t_2, \xi_{\sqrt{k}} = t_2, A_{t_2} \le k^{3/2}), \tag{4.37}
$$

where we have summed over x_1, a_1, x_2 and a_2 in $G_{k,n}$. Moreover, (4.37) yields

$$
\sum_{(\bar{t},\bar{x},\bar{a})\in G_{k,n}} S_1 S_3 \le \mathbf{P}_{\beta,\mu_\beta}(\xi_{\sqrt{k}} < \tau_0)\, \mathbf{P}_{\beta}(\xi_{\sqrt{k}} < \tau_0) \le \frac{C_3}{k}, \tag{4.38}
$$

where we have used that the probability for the V random walk to reach $\sqrt{k}$ before coming back to the lower half plane is $O(1/\sqrt{k})$. Thus,

$$
\sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \sum_{(\bar{t},\bar{x},\bar{a})\in G_{k,n}\setminus \widetilde{G}_{k,n}} S_1 \widetilde{S}_2 S_3 \le \sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \frac{C_4 \log(n)^3}{n^2 k} \le \frac{C_5 \log(n)^4}{n^2} = o(1/n^{4/3}).
$$

It remains to bound from above $\sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \sum_{(\bar{t},\bar{x},\bar{a})\in G_{k,n}\setminus \widetilde{G}_{k,n}} S_1 \widehat{S}_2 S_3$. We rewrite (4.36) as

$$
\widehat{S}_2 \le \tfrac{C_2}{n^{4/3}} \big[\tfrac{n^{2/3}}{(k-t_1-t_2)}\big]^2 e^{-\frac{3}{8\sigma_\beta^2}\left(\frac{n^{2/3}}{k-t_1-t_2}\right)^3},
$$

and we note that $x^2 e^{-3x^3/(8\sigma_\beta^2)} \le e^{-x^3/(4\sigma_\beta^2)}$ for x large enough. Since $k \le \eta n^{2/3}$ it follows that $n^{2/3}/(k - t_1 - t_2) \ge n^{2/3}/k \ge 1/\eta$ so that by choosing η small enough we get

$$
\widehat{S}_2 \le \tfrac{C_2}{n^{4/3}} e^{-\frac{1}{4\sigma_\beta^2}\left(\frac{n^{2/3}}{k-t_1-t_2}\right)^3} \le \tfrac{C_2}{n^{4/3}} e^{-\frac{1}{4\sigma_\beta^2}\left(\frac{n^{2/3}}{k}\right)^3}, \tag{4.39}
$$

and then we use (4.37) and (4.39) to get

$$\begin{aligned}\sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \sum_{(\bar t,\bar x,\bar a)\in G_{k,n}\setminus \widetilde G_{k,n}} S_1 \widehat S_2 S_3 &\le \sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \tfrac{C_2}{n^{4/3}} e^{-\frac{1}{4\sigma_\beta^2}\left(\frac{n^{2/3}}{k}\right)^3} \sum_{(\bar t,\bar x,\bar a)\in G_{k,n}\setminus \widetilde G_{k,n}} S_1 S_3\\ &\le \sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \tfrac{C_2}{k n^{4/3}} e^{-\frac{1}{4\sigma_\beta^2}\left(\frac{n^{2/3}}{k}\right)^3}\\ &\le \tfrac{C_2}{n^{4/3}}\left[\tfrac{1}{n^{2/3}} \sum_{k=n^{2/3}/\log n}^{\eta n^{2/3}} \tfrac{n^{2/3}}{k} e^{-\frac{1}{4\sigma_\beta^2}\left(\frac{n^{2/3}}{k}\right)^3}\right], \end{aligned} \tag{4.40}$$

and the Riemann sum between brackets above converges to $\int_0^\eta (1/x)e^{-1/(4\sigma_\beta^2 x^3)}dx$ so that the r.h.s. in (4.40) is smaller that $\varepsilon/n^{4/3}$ as soon as η is chosen small enough and this completes the proof.

Step 2

Our aim is to show that for all $\eta>0$,

$$\lim_{n\to\infty} n^{4/3} v_n = (1+e^{\beta/2})\sqrt{\frac{\mathbf{E}_\beta\left[V_1^2\right]}{2\pi}} \int_\eta^{+\infty} x^{-3} w(x^{-\frac{3}{2}})\,dx. \tag{4.41}$$

Proof. By Theorem 8 of [26] (see also Theorem A.11 of [19]) and since $\mathbf{E}_\beta\left[V_1^2\right]<+\infty$, we can state that $\mathbf{P}_\beta\left(\widetilde\tau=n\right)\sim Cn^{-3/2}$ with $C=(\mathbf{E}_\beta\left[V_1^2\right]/2\pi)^{1/2}$ and with $\widetilde\tau=\inf\{i\ge 1\colon V_i\le 0\}$ which may differ from τ (recall 4.1) when $V_0=0$ only. In Appendix 7.2, we extend this local limit theorem to the random walk with initial distribution μ_β and we obtain

$$\mathbf{P}_{\beta,\mu_\beta}\left(\tau=n\right)\sim C_\tau n^{-3/2} \quad\text{with}\quad C_\tau=(1+e^{\beta/2})\sqrt{\frac{\mathbf{E}_\beta\left[V_1^2\right]}{2\pi}}. \tag{4.42}$$

Let

$$v_n' := \sum_{\eta n^{2/3}\le k\le n} \mathbf{P}_{\beta,\mu_\beta}\left(\tau=k\right) k^{-3/2} w\left(\frac{n-k}{k^{3/2}}\right).$$

We recall the definition of R_n in (4.13) and we write

$$\begin{aligned}|v_n-v_n'| &\le \sum_{\eta n^{2/3}\le k\le n} \mathbf{P}_{\beta,\mu_\beta}\left(\tau=k\right) k^{-3/2} R_k \le C' R_{\eta n^{2/3}} \sum_{\eta n^{2/3}\le k\le n} k^{-3}\\ &\le C'' R_{\eta n^{2/3}} \frac{1}{\eta^2 n^{4/3}} = o(n^{-4/3}). \end{aligned} \tag{4.43}$$

We can establish by dominating convergence (see Remark 4.5) that

$$n^{4/3} v_n' \to C_\tau \int_\eta^{+\infty} t^{-3} w(t^{-3/2})\,dt. \tag{4.44}$$

By putting together (4.43), (4.44) we obtain (4.41) and this completes the proof. □

4.3 Proof of Theorem 2.2 (2)

Let $(\mathfrak{U}_i)_{i=1}^\infty$ be the sequence of inter-arrivals of a $1/3$-stable regenerative set [1] $\mathfrak{T}$ on $[0,1]$, conditioned on $1\in\mathfrak{T}$ and denote by $(\mathfrak{U}^i)_{i=0}^\infty$ its order statistics. Let $(Y_i)_{i=1}^\infty$ be an

[1] We refer to [9, Appendix A] for a self-contained introduction of the α-stable regenerative sets on $[0,1]$ (see also [2]). In fact, it is useful to keep in mind that such a set is the limit in distribution of the set $\frac{\tau}{N}\cap[0,1]$ when τ is a regenerative process on $\mathbb{N}$ with an inter-arrival law K that satisfies $K(n)\sim L(n)/n^{1+\alpha}$ and with L a slowly varying function. The alpha-stable regenerative set can also be viewed as the zero set of a Bessel bridge on $[0,1]$ of dimension $d=2(1-\alpha)$.

IID sequence of continuous random variables, independent of $(\mathfrak{U}_i)_{i=0}^{\infty}$, with density

$$\mathbf{d}\mathbf{P}_{Y_1}(x) \propto \tfrac{1}{x^3}\, w\big(\tfrac{1}{x^{3/2}}\big) 1_{\mathbb{R}^+}(x). \tag{4.45}$$

We are first going to prove that

$$\lim_{L\to+\infty} \frac{N_l}{L^{2/3}} =_{law} \sum_{i=1}^{+\infty} Y_i\, (\mathfrak{U}^i)^{2/3} =_{law} \sum_{i=1}^{+\infty} Y_i\, \mathfrak{U}_i^{2/3}\,, \tag{4.46}$$

where the second identity in law in (4.46) is obvious and then we shall identify the distribution of $\sum_{i=1}^{+\infty} Y_i\, \mathfrak{U}_i^{2/3}$ with the distribution of g_1 conditionally on $B_{g_1} = 0$.

We recall (4.1–4.3) and we consider the i.i.d. sequence of random vectors $(\mathfrak{N}_i, \mathfrak{A}_i)_{i=1}^{\infty}$ and we recall that $X_i = \mathfrak{N}_i + \mathfrak{A}_i$ for $i \in \mathbb{N}$. We recall that, under $\mathbf{P}_\beta$, the first excursion has law $\mathbf{P}_{\beta,0}$ and the next excursions have law $\mathbf{P}_{\beta,\mu_\beta}$. Let us set $S_n = X_1 + \cdots + X_n$ and $v_L := \max\{i \geq 0\colon S_i \leq L\}$. We recall (4.17) and we consider the sequence

$$(\mathfrak{N}_i, \mathfrak{A}_i, X_i)_{i=1}^{v_L}$$

under the law $P_\beta(\cdot | L \in \mathfrak{X})$. We denote by $X_{r_1} \geq \cdots \geq X_{r_{v_L}}$ the order statistics of $(X_i)_{i=1}^{v_L}$ such that if $X_{r_i} = X_{r_j}$ and $i < j$ then $r_i < r_j$. To simplify notations we set $(\mathfrak{N}^i, \mathfrak{A}^i, X^i) = (\mathfrak{N}_{r_i}, \mathfrak{A}_{r_i}, X_{r_i})$ for $i \in \{1, \ldots, v_L\}$.

To begin with, we will prove (4.46) subject to Propositions 4.8 and Claim 4.9 below. Then, the remainder of this section will be dedicated to the proof of Propositions 4.8 and Claim 4.9.

Proposition 4.8.

$$\lim_{L\to\infty} \frac{\sum_{i=1}^{v_L} \mathfrak{N}_i}{L^{2/3}} =_{\text{Law}} \sum_{i=1}^{\infty} Y_i\, (\mathfrak{U}^i)^{2/3}. \tag{4.47}$$

Claim 4.9. *For $\beta = \beta_c$, $t \in [0, \infty)$ and L large enough,*

$$P_{L,\beta}\bigg(\frac{N_l}{L^{2/3}} \leq t\bigg) = \sum_{r=0}^{tL^{2/3}-1} \xi_{r,L}\, \mathbf{P}_\beta\bigg(\frac{r + \sum_{i=1}^{v_{L-r+1}} \mathfrak{N}_i}{L^{2/3}} \leq t \,|\, L - r + 1 \in \mathfrak{X}\bigg) \tag{4.48}$$

with

$$\xi_{r,L} = \frac{(1 - e^{-\beta/2}) \mathbf{P}_\beta\, (L - r + 1 \in \mathfrak{X})}{c_\beta^{r-1}\, \widetilde{Z}_{L,\beta}}, \quad r \in \{0, \ldots, L\}. \tag{4.49}$$

Pick $t \in [0, \infty)$ and $\varepsilon > 0$. Combining (4.49) with (4.16) and Theorem 2.1 (2), we obtain that

$$\sum_{r=1}^{L-1} \xi_{r,L} = 1 + o(1). \tag{4.50}$$

Moreover, by combining (4.49) with (4.18) and Theorem 2.1 (2) we can claim that there exists an $r_\varepsilon \in \mathbb{N}$ such that, provided L is chosen large enough, we have $\sum_{r \geq r_\varepsilon} \xi_{r,L} \leq \varepsilon$. Thus, with (4.50) we have also that $\sum_{r=0}^{r_\varepsilon} \xi_{r,L} \in [1 - \varepsilon, 1]$. Then, we use Claim 4.9 and we apply Proposition 4.8 to each probability indexed by $r \in \{1, \ldots, r_\varepsilon\}$ in the r.h.s. of (4.48) to conclude that, for L large enough

$$(1-\varepsilon) P\bigg(\sum_{i=1}^{\infty} Y_i\, (\mathfrak{U}^i)^{2/3} \leq t\bigg) - \varepsilon \leq P_{L,\beta}\Big(\frac{N_l}{L^{2/3}} \leq t\Big) \leq 2\varepsilon + P\bigg(\sum_{i=1}^{\infty} Y_i\, (\mathfrak{U}^i)^{2/3} \leq t\bigg), \tag{4.51}$$

which completes the proof of (4.46).

Proof of Proposition 4.8

To begin with, let us distinguish between the k excursions associated with the first k variables of the order statistics $(X^i)_{i=1}^{v_L}$ and the others, i.e.,

$$\frac{\sum_{i=1}^{v_L} \mathfrak{N}_i}{L^{2/3}} = A_{k,L} + B_{k,L} \tag{4.52}$$

with

$$A_{k,L} = \sum_{i=1}^{k} \frac{\mathfrak{N}^i}{L^{\frac{2}{3}}} \quad \text{and} \quad B_{k,L} = \sum_{i=k+1}^{v_L} \frac{\mathfrak{N}^i}{(X^i)^{\frac{2}{3}}} \Big(\frac{X^i}{L}\Big)^{\frac{2}{3}}. \tag{4.53}$$

Then, the proof of Proposition 4.8 will be deduced from the following two steps.

Step 1 Show that for all $k \in \mathbb{N}$ and under $\mathbf{P}_\beta(\cdot \,|\, L \in \mathfrak{X})$,

$$\lim_{L\to\infty} A_{k,L} =_{\text{law}} \textstyle\sum_{i=1}^{k} Y_i\, (\mathfrak{U}^i)^{\frac{2}{3}}. \tag{4.54}$$

Step 2 Show that for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{L\to\infty} \mathbf{P}_\beta(B_{k,L} \geq \varepsilon | L \in \mathfrak{X}) = 0. \tag{4.55}$$

Before proving (4.54) and (4.55), we need to settle some preparatory lemmas. To begin with we let F be the distribution function of X under $\mathbf{P}_{\beta,\mu_\beta}$ that is $F(t) = \mathbf{P}_{\beta,\mu_\beta}(X \leq t)$ for $t \in \mathbb{R}$ and F^{-1} its pseudo-inverse, that is $F^{-1}(u) = \inf\{t \in \mathbb{R} \colon\ F(t) \geq u\}$ for $u \in (0,1)$.

Lemma 4.10. *There exists $C > 0$ such that*

$$F^{-1}(u) \sim \tfrac{C}{(1-u)^3} \quad \text{as } u \to 1^-. \tag{4.56}$$

Proof. The proof is a straightforward consequence of (4.18). □

Recall (4.45). The next lemma deals with the convergence in law, as $m \to \infty$, of the horizontal extension of an excursion renormalized by $m^{2/3}$ and conditioned on the area of the excursion being equal to m.

Lemma 4.11. *For all $\beta > 0$ and all $m \in \mathbb{N}$ we consider the random variable $\frac{\mathfrak{N}}{m^{2/3}}$ under the laws $\mathbf{P}_{\beta,a}(\cdot|X = m)$ with $a \in \{0, \mu_\beta\}$. We have*

$$\lim_{m\to\infty} \frac{\mathfrak{N}_1}{m^{2/3}} =_{Law} Y_1, \tag{4.57}$$

and also that the sequence $\big(\mathbf{E}_{\beta,a}\big(\frac{\mathfrak{N}}{m^{2/3}}\big| X = m\big)\big)_{m\in\mathbb{N}}$ is bounded.

Proof. With the help of Theorem (4.4) we can use the following equality

$$P_{\beta,\mu_\beta}(\mathfrak{N} = n \,\big|\, X = m) = P_{\beta,\mu_\beta}(\mathfrak{A} = m - n \,\big|\, \mathfrak{N} = n)\ \frac{P_{\beta,\mu_\beta}(\mathfrak{N} = n)}{P_{\beta,\mu_\beta}(X = m)}, \tag{4.58}$$

combined with (4.42) and (4.18), to claim that there exists a $D > 0$ such that

$$P_{\beta,\mu_\beta}(\mathfrak{N} = n \,\big|\, X = m) = D\frac{m^{4/3}}{n^3} w\Big(\frac{m-n}{n^{3/2}}\Big) + \frac{m^{4/3}}{n^3}\left(\varepsilon_1(m) + \varepsilon_2(n)\right), \tag{4.59}$$

with $\varepsilon_1(m)$ and $\varepsilon_2(n)$ vanishing as $m, n \to \infty$.

To display an upper bound for the sequence $\big(\mathbf{E}_{\beta,\mu_\beta}\big(\frac{\mathfrak{N}}{m^{2/3}}\big|X=m\big)\big)_{m\in\mathbb{N}}$ it suffices of course to consider

$$\mathbf{E}_{\beta,\mu_\beta}\Big(\frac{\mathfrak{N}}{m^{2/3}}1_{\{\mathfrak{N}\geq m^{2/3}\}}\big|X=m\Big)=\sum_{n=m^{2/3}}^{m}\frac{n}{m^{2/3}}P_{\beta,\mu_\beta}(\mathfrak{N}=n\,\big|\,X=m) \quad (4.60)$$
$$=\sum_{n=m^{2/3}}^{m}D\frac{m^{2/3}}{n^2}w\Big(\frac{m-n}{n^{3/2}}\Big)+\sum_{n=m^{2/3}}^{m}\frac{m^{2/3}}{n^2}(\varepsilon_1(m)+\varepsilon_2(n)),$$

where we have used (4.59). Since the second term in the r.h.s. in (4.60) clearly vanishes as $m\to\infty$, we focus on the first term and since w is uniformly continuous because $s\to w(s)$ is continuous on $[0,\infty)$ and vanishes as $s\to\infty$, we can write the first term as a Riemann sum that converges to $\int_1^\infty\frac{D}{x^2}\,w(\frac{1}{x^{3/2}})dx$ plus a rest that vanishes as $m\to\infty$ and this gives us the expected boundedness.

Similarly, the convergence in law is obtained by picking $t\in[0,\infty)$ and by writing $\mathbf{P}_{\beta,\mu_\beta}\Big(\frac{\mathfrak{N}_1}{m^{2/3}}\leq t\,\big|\,X=m\Big)=:\widetilde{u}_m+\widetilde{v}_m$ where

$$\widetilde{u}_m=\frac{1}{\mathbf{P}_{\beta,\mu_\beta}(X=m)}\sum_{k=1}^{\eta\,m^{2/3}}\mathbf{P}_{\beta,\mu_\beta}\left(A_{k-1}=m-k,\tau=k\right) \quad (4.61)$$

$$\widetilde{v}_m=\frac{1}{\mathbf{P}_{\beta,\mu_\beta}(X=m)}\sum_{k=\eta\,m^{2/3}}^{tm^{2/3}}\mathbf{P}_{\beta,\mu_\beta}\left(A_{k-1}=m-k,\tau=k\right), \quad (4.62)$$

where $\eta\in(0,t)$. We note easily that $\widetilde{u}_m=[(1-e^{-\beta/2})\,\mathbf{P}_{\beta,\mu_\beta}(X=m)]^{-1}\,u_m$ with u_m defined in (4.21). Therefore, (4.18) and (4.22) tell us that $\widetilde{u}_m$ can be made arbitrarily small provided η is small enough and m large enough. Thus, it remains to deal with $\widetilde{v}_m$, which, with the help of (4.18) is treated as the second term in the r.h.s. in (4.21). Thus, (4.41) tells us that there exists a $D>0$ such that $\lim_{m\to\infty}\widetilde{v}_m=\int_\eta^t Dt^{-3}w(t^{-3/2})\,dt$ and this suffices to complete the proof of (4.57). □

Lemma 4.12. *For $\beta>0$ and $\varepsilon>0$, there exists a $c_\varepsilon>0$ such that for $L\in\mathbb{N}$,*

$$\mathbf{P}_\beta(v_L\geq c_\varepsilon\,L^{1/3})\leq\varepsilon. \quad (4.63)$$

Proof. Since under $\mathbf{P}_\beta$ only the first excursion has law $\mathbf{P}_{\beta,0}$ and the others $\mathbf{P}_{\beta,\mu_\beta}$ the proof of (4.63) will be complete once we show for instance that for c large enough and $L\in\mathbb{N}$

$$\mathbf{P}_{\beta,\mu_\beta}(\max\{X_i,i\leq cL^{1/3}\}\leq L)\leq\varepsilon. \quad (4.64)$$

We recall that, if $(\Gamma_i)_{i=1}^{cL^{1/3}+1}$ are the partial sums of $(\gamma_i)_{i=1}^{cL^{1/3}+1}$ a sequence of IID exponential random variables with parameter 1, we can state that

$$\max\{X_i,i\leq cL^{1/3}\}=_{\text{law}}\;F^{-1}\big(\,\widetilde{\Gamma}_{cL^{1/3}}/\Gamma_{cL^{1/3}+1}\big). \quad (4.65)$$

with $\widetilde{\Gamma}_{cL^{1/3}}=\gamma_2+\cdots+\gamma_{cL^{1/3}+1}$. Thus we can rewrite the l.h.s. in (4.64) as

$$\mathbf{P}\bigg(F^{-1}\Big(\frac{\widetilde{\Gamma}_{cL^{1/3}}}{\Gamma_{cL^{1/3}+1}}\Big)\leq L\bigg)=\mathbf{P}\big(F^{-1}(D_L)(1-D_L)^3\leq L(1-D_L)^3\big), \quad (4.66)$$

with $D_L=\widetilde{\Gamma}_{cL^{1/3}}/\Gamma_{cL^{1/3}+1}$. After some easy simplifications we rewrite (4.66) as

$$\mathbf{P}\bigg(\gamma_1\geq\big[F^{-1}(D_L)\big]^{\frac{1}{3}}(1-D_L)\;\frac{\Gamma_{cL^{1/3}+1}}{L^{\frac{1}{3}}}\bigg), \quad (4.67)$$

and then we use the law of large number combined with Lemma 4.10 to claim that, as $L \to \infty$, the r.h.s. in (4.67) converges to $P(\gamma_1 \geq c\, C^{1/3})$, which can be made arbitrarily small for c large (note that C is the positive constant appearing in Lemma 4.10). This completes the proof of the lemma. □

Lemma 4.13. *For every $\beta > 0$, there exists a $M > 0$ such that, for every function $G : \mathcal{P}(\{0, \dots, L/2\}) \to \mathbb{R}^+$, we have*

$$\mathbf{E}_\beta\Big[G\big(\mathfrak{X} \cap [0, \tfrac{L}{2}]\big) \,\big|\, L \in \mathfrak{X}\Big] \leq M\, \mathbf{E}_\beta\Big[G\big(\mathfrak{X} \cap [0, \tfrac{L}{2}]\big)\Big]. \tag{4.68}$$

Proof. We compute the Radon Nikodym density of the image measure of $\mathbf{P}_\beta(\cdot | L \in \mathfrak{X})$ by $\mathfrak{X} \cap [0, L/2]$ w.r.t. its counterpart without conditioning. For $0 = y_0 < y_1 < \dots < y_m \leq L/2$ we obtain

$$\frac{\mathbf{P}_\beta(\tau \cap [0, \frac{L}{2}] = (y_0, \dots, y_m) | L \in \mathfrak{X})}{\mathbf{P}_\beta(\tau \cap [0, \frac{L}{2}] = (y_0, \dots, y_m))} := G_L(y_m) + K_L(y_m),$$

with

$$\begin{aligned} G_L(y) &= \frac{\sum_{n=0}^{L/4} \mathbf{P}_{\beta,\mu_\beta}(n \in \mathfrak{X})\ \ \mathbf{P}_{\beta,\mu_\beta}(X = L - n - y)}{\mathbf{P}_\beta(L \in \mathfrak{X})\ P_{\beta,\mu_\beta}(X \geq \frac{L}{2} - y)}, \\ K_L(y) &= \frac{\sum_{n=L/4}^{L/2} \mathbf{P}_{\beta,\mu_\beta}(n \in \mathfrak{X})\ \ \mathbf{P}_{\beta,\mu_\beta}(X = L - n - y)}{\mathbf{P}_\beta(L \in \mathfrak{X})\ \mathbf{P}_{\beta,\mu_\beta}(X \geq \frac{L}{2} - y)}. \end{aligned} \tag{4.69}$$

Note that, for $y = 0$, the terms $\mathbf{P}_{\beta,\mu_\beta}(X = L - n - y)$ and $\mathbf{P}_{\beta,\mu_\beta}(X \geq \frac{L}{2} - y)$ in the expression of $G_L(y)$ and $K_L(y)$ should be replaced by $\mathbf{P}_{\beta,0}(X = L - n - y)$ and $\mathbf{P}_{\beta,0}(X \geq \frac{L}{2} - y)$, respectively. However, this does not change anything in the sequel and this is why we will focus on $y > 0$. It remains to prove that $G_L(y)$ and $K_L(y)$ are bounded above uniformly in $L \in \mathbb{N}$ and $y \in \{0, \dots, L/2\}$. We will focus on G_L since K_L can be treated similarly.

The constants $c_1, \dots, c_4$ below are positive and independent of L, n, y. By recalling (4.18) and since $L - n - y \geq L/4$ when $n \in \{0, \dots, L/4\}$ we can claim that in the numerator of $G_L(y)$, the term $P_{\beta,\mu_\beta}(X = L - n - y)$ is bounded above by $c_1/L^{4/3}$ independently of n while (4.19) implies that $\sum_{n=0}^{L/4} P_{\beta,\mu_\beta}(n \in \mathfrak{X}) \leq c_2 L^{1/3}$. Let us now deal with the denominator: (4.19) tell us that $\mathbf{P}_{\beta,\mu_\beta}(L \in \mathfrak{X}) \geq c_3/L^{2/3}$ while (4.18) gives

$$\mathbf{P}_{\beta,\mu_\beta}(X \geq \tfrac{L}{2} - y) \geq \mathbf{P}_{\beta,\mu_\beta}(X \geq \tfrac{L}{2}) \geq \tfrac{c_4}{L^{1/3}}.$$

As a consequence, $G_L(y)$ is bounded above uniformly in $L \in \mathbb{N}$ and $y \in \{0, \dots, L/2\}$. □

We resume the proof of Proposition 4.8.

Proof of Step 1 (4.54)

The proof of Step 1 will be complete once we show that

$$\lim_{l \to \infty} \left(\Big(\frac{X^1}{L}\Big)^{\frac{2}{3}}, \dots, \Big(\frac{X^k}{L}\Big)^{\frac{2}{3}}, \frac{\mathfrak{N}^1}{(X^1)^{\frac{2}{3}}}, \dots, \frac{\mathfrak{N}^k}{(X^k)^{\frac{2}{3}}} \right) =_{\text{Law}} (\mathfrak{U}^1, \dots, \mathfrak{U}^k, Y_1, \dots, Y_k). \tag{4.70}$$

To obtain this convergence in law, we consider $g_1, \dots, g_k$ that are real Borel and bounded functions. We consider also $t \in \mathbb{N}$ and $(x_i)_{i=1}^t$ a sequence of strictly positive integers satisfying $x_1 + \dots + x_t = L$ with an order statistics $x_{r_1} \geq \dots \geq x_{r_t}$. The key observation is that, by independence of the $(\mathfrak{N}_i, X_i)_{i \in \mathbb{N}}$, we have

$$\mathbf{E}_\beta\Bigg[\prod_{j=1}^k g_j\Big(\tfrac{\mathfrak{N}^j}{(X^j)^{\frac{2}{3}}}\Big) \,\Big|\, X_i = x_i,\, 1 \leq i \leq t\Bigg] = \prod_{j=1}^k \mathbf{E}_{\beta,\, 0\, 1_{\{r_j = 1\}} + \mu_\beta\, 1_{\{r_j > 1\}}}\Bigg[g_j\Big(\tfrac{\mathfrak{N}}{X^{\frac{2}{3}}}\Big) \,\Big|\, X = x_{r_j}\Bigg]. \tag{4.71}$$

We consider $f_1,\dots,f_k,g_1,\dots,g_k$ that are real Borelian and bounded functions and we use (4.71) to observe that

$$\begin{aligned}&\mathbf{E}_\beta\Big[\prod_{j=1}^k f_j\Big(\big[\tfrac{X^j}{L}\big]^{\frac23}\Big)\, g_j\Big(\tfrac{\mathfrak{N}^j}{(X^j)^{\frac23}}\Big)\,\Big|\, L\in\mathfrak{X}\Big] \\ &\quad = \mathbf{E}_\beta\Big[\prod_{j=1}^k f_j\Big(\big[\tfrac{X^j}{L}\big]^{\frac23}\Big)\,\mathbf{E}_{\beta,\,0\,1_{\{r_j=1\}}+\mu_\beta\,1_{\{r_j>1\}}}\Big[g_j\Big(\tfrac{\mathfrak{N}}{X^{\frac23}}\Big)\Big|\, X=X^j\Big]\,\Big|\, L\in\mathfrak{X}\Big].\end{aligned}\tag{4.72}$$

Because of Lemma 4.6, we can assert that, under $\mathbf{P}_\beta(\cdot|L\in\mathfrak{X})$ the random set $(\mathfrak{X}/L)\cap[0,1]$ converges in law towards $\mathfrak{U}$, i.e., the $1/3$ regenerative set on $[0,1]$ conditioned on $1\in\mathfrak{U}$ (the latter convergence is proven e.g. in [9], Proposition A.8). As a consequence $\big((\frac{X^1}{L}),\dots,(\frac{X^k}{L})\big)$ converges in law towards $(\mathfrak{U}^1,\dots,\mathfrak{U}^k)$ which implies that $X^1,\dots,X^k$ tend to ∞ in probability and therefore we can use Lemma 4.11 to show that the l.h.s. in (4.72) tends to (as $L\to\infty$)

$$\prod_{j=1}^k \mathbf{E}\big[g_j(Y)\big]\mathbf{E}\Big[\prod_{j=1}^k f_j(\mathfrak{U}^j)\Big].\tag{4.73}$$

Thus, the proof of Step 1 is complete.

Proof of Step 2 (4.55)

One easily check that, under $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L\in\mathfrak{X})$, the following reversibility holds true, i.e.,

$$(\mathfrak{N}_i,\mathfrak{A}_i,X_i)_{i=1}^{v_L} =_{\text{law}} (\mathfrak{N}_{1+v_L-i},\mathfrak{A}_{1+v_L-i},X_{1+v_L-i})_{i=1}^{v_L}.\tag{4.74}$$

However, (4.74) is not true under $\mathbf{P}_\beta(\cdot|L\in\mathfrak{X})$ since $(\mathfrak{N}_1,\mathfrak{A}_1,X_1)$ does not have the same law has its counterparts indexed in $\mathbb{N}\setminus\{1\}$. For this reason, in the first part of the proof we will show that proving (4.55) under $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L\in\mathfrak{X})$ yields that (4.55) also holds true under $\mathbf{P}_\beta(\cdot|L\in\mathfrak{X})$ and then, in the second part of the proof, we will show that (4.55) is true under $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L\in\mathfrak{X})$.

Part 1: (4.55) **under** $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L\in\mathfrak{X})$ **yields** (4.55) **under** $\mathbf{P}_\beta(\cdot|L\in\mathfrak{X})$**.** Let us first note that if $V:=(V_i)_{i=0}^\infty$ is a random walk of law $\mathbf{P}_\beta$, then $\widetilde V:=(V_{i+1})_{i=0}^\infty$ is random walk of law $\mathbf{P}_{\beta,\nu_\beta}$ where ν_β is the law of an increment of the random walk (recall (1.6)), i.e.,

$$\nu_\beta(k)=\frac{e^{-\frac{\beta}{2}|k|}}{c_\beta},\quad k\in\mathbb{Z}.\tag{4.75}$$

The Radon-Nikodym density of $\mathbf{P}_{\beta,\nu_\beta}$ with respect to $\mathbf{P}_{\beta,\mu_\beta}$ is a function of V_0, i.e.,

$$\frac{\mathrm{d}\mathbf{P}_{\beta,\nu_\beta}}{\mathrm{d}\mathbf{P}_{\beta,\mu_\beta}}(V)=\frac{2}{1+e^{-\beta}}\,1_{\{V_0\neq0\}}+\frac{1}{1+e^{-\beta}}\,1_{\{V_0=0\}},\quad V=(V_i)_{i=0}^\infty\in\mathbb{Z}^{N_0},\tag{4.76}$$

and therefore, it is bounded above and below by two positive constants. Consequently, proving (4.55) under $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L\in\mathfrak{X})$ or under $\mathbf{P}_{\beta,\nu_\beta}(\cdot|L\in\mathfrak{X})$ is equivalent. As a consequence, the first part of the proof will be complete once we show that (4.55) under $\mathbf{P}_{\beta,\nu_\beta}(\cdot|L\in\mathfrak{X})$ yields (4.55) under $\mathbf{P}_\beta(\cdot|L\in\mathfrak{X})$. To that purpose, we consider $V=(V_i)_{i=0}^\infty$ a random walk of law $\mathbf{P}_\beta$ and we recall that $\widetilde V:=(V_{i+1})_{i=0}^\infty$ is a random walk of law $\mathbf{P}_{\beta,\nu_\beta}$. It turns out in this case that

$$(\mathfrak{N}_i,\mathfrak{A}_i,X_i)_{i=1}^\infty=\big(1_{\{i=1\}}+\widetilde{\mathfrak{N}}_i,\widetilde{\mathfrak{A}}_i,1_{\{i=1\}}+\widetilde X_i\big)_{i=1}^\infty,\tag{4.77}$$

where the sequence $(\widetilde{\mathfrak{N}}_i, \widetilde{\mathfrak{A}}_i, \widetilde{X}_i)_{i=1}^{\infty}$ is defined with $\widetilde{V}$ as in (4.1-4.3) and (4.17) while $(\mathfrak{N}_i, \mathfrak{A}_i, X_i)_{i=1}^{\infty}$ is defined with V. Obviously, (4.77) yields that $v_L = \widetilde{v}_{L-1}$ and that $\{L \in \mathfrak{X}\} = \{L-1 \in \widetilde{\mathfrak{X}}\}$. We let $\widetilde{X}_{\widetilde{r}_1} \geq \cdots \geq \widetilde{X}_{\widetilde{r}_{v_L}}$ be the order statistics of $(\widetilde{X}_i)_{i=1}^{v_L}$ and we can write

$$\widetilde{B}_{k-1,L-1} := \sum_{i=k}^{\widetilde{v}_{L-1}} \frac{\widetilde{\mathfrak{N}}_{\widetilde{r}_i}}{(L-1)^{2/3}} \geq \sum_{i=k}^{v_L} \frac{\mathfrak{N}_{\widetilde{r}_i}}{L^{2/3}} - \frac{1}{(L-1)^{2/3}}, \tag{4.78}$$

where, in the last inequality of (4.78), the term $1/(L-1)^{2/3}$ is subtracted to take into account the fact that $\widetilde{\mathfrak{N}}_1 = \mathfrak{N}_1 - 1$ (recall (4.77)) which plays a role if $1 \notin \{\widetilde{r}_1, \dots, \widetilde{r}_{k-1}\}$. At this stage, it remains to note that if $1 \notin \{r_1, \dots, r_k\}$ or if $1 \in \{r_1, \dots, r_k\}$ and $X_1 > X_{r_k}$ then $\{r_1, \dots, r_k\} = \{\widetilde{r}_1, \dots, \widetilde{r}_k\}$. Otherwise, if $1 \in \{r_1, \dots, r_k\}$ and $X_1 = X_{r_k}$ then $\{r_1, \dots, r_k\} \setminus \{1\} = \{\widetilde{r}_1, \dots, \widetilde{r}_{k-1}\}$ so that in any case (4.78) yields

$$\widetilde{B}_{k-1,L-1} \geq \sum_{i=k+1}^{v_L} \frac{\mathfrak{N}_{r_i}}{(L-1)^{2/3}} - \frac{1}{(L-1)^{2/3}} \geq B_{k,L} - \frac{1}{(L-1)^{2/3}},$$

which is sufficient to conclude that (4.55) under $\mathbf{P}_{\beta,\nu_\beta}(\cdot|L \in \mathfrak{X})$ yields (4.55) under $\mathbf{P}_\beta(\cdot|L \in \mathfrak{X})$.

Part 2: proof of (4.55) **under** $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L \in \mathfrak{X})$**.** Reversibility yields that, under $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L \in \mathfrak{X})$,

$$(\mathfrak{N}_i, \mathfrak{A}_i, X_i)_{i=1}^{v_L} =_{\text{law}} (\mathfrak{N}_{1+v_L-i}, \mathfrak{A}_{1+v_L-i}, X_{1+v_L-i})_{i=1}^{v_L}.$$

We set $v_{L/2} := \max\{i \geq 1 \colon\ S_i \leq L/2\}$ and $v'_{L/2} := v_L - \min\{i \geq 1 \colon\ S_i \geq L/2\}$ such that $v_{L/2}$ and $v'_{L/2}$ have the same law and

$$(\mathfrak{N}_i, \mathfrak{A}_i, X_i)_{i=1}^{v_{L/2}} =_{\text{law}} (\mathfrak{N}_{1+v_L-i}, \mathfrak{A}_{1+v_L-i}, X_{1+v_L-i})_{i=1}^{v'_{L/2}}. \tag{4.79}$$

We also denote by $(\mathfrak{N}_{\text{mid}}, \mathfrak{A}_{\text{mid}}, X_{\text{mid}})$ the features of the excursion containing $L/2$ in case $\tau_{v_{L/2}} < L/2$. In case $\tau_{v_{L/2}} = L/2$, we set $(\mathfrak{N}_{\text{mid}}, \mathfrak{A}_{\text{mid}}, X_{\text{mid}}) = (0,0,0)$.

By applying Lemmas 4.12 and 4.13 and time reversibility we can state that by choosing c large enough, the quantity $\mathbf{P}_{\beta,\mu_\beta}\left(v_{L/2}, v'_{L/2} \leq cL^{1/3} | L \in \mathfrak{X}\right)$ is arbitrary close to 1 uniformly in L. Thus, we set

$$H_{c,L} = \{L \in \mathfrak{X}\} \cap \{v_L, v'_L \leq cL^{1/3}\},$$

and Step 2 will be complete once we show that, for each $c > 0$ and $\varepsilon > 0$.

$$\lim_{k\to\infty} \limsup_{L\to\infty} \mathbf{P}_{\beta,\mu_\beta}\left(B_{k,L} \geq \varepsilon | H_{c,L}\right) = 0. \tag{4.80}$$

By Markov's inequality Step 2 will be a consequence of

$$\lim_{k\to\infty} \limsup_{L\to\infty} \mathbf{E}_{\beta,\mu_\beta}\left[B_{k,L} | H_{c,L}\right] = 0. \tag{4.81}$$

We recall (4.53) and we compute $\mathbf{E}_\beta(B_{k,L}|H_{c,L})$ by conditioning on $\sigma(X_i, i \in \mathbb{N})$ as we did in (4.71). We recall that $H_{c,L}$ is $\sigma(X_i, i \in \mathbb{N})$-measurable.

$$\mathbf{E}_\beta(B_{k,L}|H_{c,L}) = \mathbf{E}_{\beta,\mu_\beta}\left[\sum_{i\geq k+1}^{v_L} \mathbf{E}_{\beta,\mu_\beta}\left[\frac{\mathfrak{N}}{X^{\frac{2}{3}}}\Big| X = X^i\right] \left(\frac{X^i}{L}\right)^{\frac{2}{3}} \Bigg| H_{c,L}\right], \tag{4.82}$$

but then, we can use Lemma 4.11 which allows us to bound by $M > 0$ each term $\mathbf{E}_{\beta,a}\left[\mathfrak{N}/X^{\frac{2}{3}} \big| X = X_i\right]$ with $a \in \{0, \mu_\beta\}$. By using again the fact there exists an $\eta > 0$ such that $\mathbf{P}_{\beta,\mu_\beta}(H_{c,L}|L\in\mathfrak{X})\geq\eta$ uniformly in L we can claim that the proof will be complete once we show that

$$\lim_{k\to\infty}\limsup_{L\to\infty}\mathbf{E}_{\beta,\mu_\beta}\left[\sum_{i\geq k+1}^{v_L}\Big(\frac{X^i}{L}\Big)^{\frac{2}{3}}\,1_{\{v_{L/2}\leq cL^{1/3}\}}1_{\{v'_{L/2}\leq cL^{1/3}\}}\Big|L\in\mathfrak{X}\right]=0. \tag{4.83}$$

We note that, under $\mathbf{P}_{\beta,\mu_\beta}(\cdot|L\in\mathfrak{X})$ we have necessarily $X^i\leq L/k$ for $i\geq k+1$. For simplicity we assume that $k\in 2\mathbb{N}$ and we denote by $(\widetilde{X}^1,\dots,\widetilde{X}^{v_{L/2}})$ the order statistics of the variables $(X_1,\dots,X_{v_{L/2}})$ and by $(\bar{X}^1,\dots,\bar{X}^{v'_{L/2}})$ the order statistics of the variables $(X_{v_L},X_{v_L-1},\dots,X_{1+v_L-v'_{L/2}})$. Then, we can easily note that $\sum_{i=1}^k X^i\geq\sum_{i=1}^{k/2}\widetilde{X}^i+\bar{X}^i$ so that the expectation in (4.83) is bounded above by

$$2\mathbf{E}_{\beta,\mu_\beta}\left[\sum_{i=k/2+1}^{v_{L/2}}\left(\frac{\widetilde{X}^i}{L}\right)^{\frac{2}{3}}1_{\{v_{L/2}\leq cL^{1/3}\}}\Big|L\in\mathfrak{X}\right]+\mathbf{E}_{\beta,\mu_\beta}\left[\left(\frac{X_{\text{mid}}}{L}\right)^{\frac{2}{3}}1_{\{X_{\text{mid}}\leq L/k\}}\Big|L\in\mathfrak{X}\right], \tag{4.84}$$

where the factor 2 in front of the first term is a direct consequence of (4.79). The second term in (4.84) is clearly bounded by $(1/k)^{2/3}$ and therefore, it can be omitted. As a consequence, it suffices to show that

$$\lim_{k\to\infty}\limsup_{L\to\infty}\mathbf{E}_{\beta,\mu_\beta}\left[\sum_{i=k}^{v_{L/2}}\Big(\frac{\widetilde{X}^i}{L}\Big)^{\frac{2}{3}}1_{\{v_{L/2}\leq cL^{1/3}\}}\Big|L\in\mathfrak{X}\right]=0. \tag{4.85}$$

At this stage, we note that $\sum_{i=k}^{v_{L/2}}(\frac{\widetilde{X}^i}{L})^{\frac{2}{3}}1_{\{v_{L/2}\leq cL^{1/3}\}}$ only depends on the random set of points $\mathfrak{X}\cap[0,L/2]$ and this allows us to use Lemma 4.13 to claim that proving (4.85) without the conditioning by $\{L\in\mathfrak{X}\}$ is sufficient. Therefore, we only need to estimate the quantity

$$\mathbf{E}_{\beta,\mu_\beta}\left[\sum_{i=k}^{cL^{1/3}}\Big(\frac{X^i}{L}\Big)^{\frac{2}{3}}\right],$$

where $(X^1,\dots,X^{cL^{1/3}})$ is the order statistics of $(X_1,\dots,X_{cL^{1/3}})$ under $\mathbf{P}_\beta$ without any conditioning. We recall that, if $(\Gamma_i)_{i=1}^{cL^{1/3}+1}$ are the partial sums of $(\gamma_i)_{i=1}^{cL^{1/3}+1}$ a sequence of IID exponential random variables with parameter 1, then it is a standard fact that

$$(X^i)_{i=1}^{cL^{1/3}}=_{\text{law}}\left(F^{-1}\left[\frac{\Gamma_{cL^{1/3}+1-i}}{\Gamma_{cL^{1/3}+1}}\right]\right)_{i=1}^{cL^{1/3}}. \tag{4.86}$$

Moreover, by Lemma 4.10, we can claim that there exists a $M>0$ such that $F^{-1}(u)\leq M/(1-u)^3$ for all $u\in(0,1)$ and consequently

$$\mathbf{E}_{\beta,\mu_\beta}\left[\sum_{i=k}^{cL^{1/3}}\Big(\frac{X^i}{L}\Big)^{\frac{2}{3}}\right]\leq\frac{M^{2/3}}{L^{2/3}}\sum_{i=k}^{cL^{1/3}}\mathbf{E}\bigg(\left[\frac{\Gamma_{cL^{1/3}+1}}{\Gamma_i}\right]^2\bigg), \tag{4.87}$$

but then we can bound from above the general term in the sum of the r.h.s. in (4.87) by

$$\begin{aligned}\mathbf{E}\bigg(\left[\frac{\Gamma_{cL^{1/3}+1}}{\Gamma_i}\right]^2\bigg)&=c^2L^{\frac{2}{3}}\mathbf{E}\bigg(\left[\frac{\Gamma_{cL^{1/3}+1}}{cL^{1/3}}\right]^2\left[\frac{1}{\Gamma_i}\right]^2\bigg)\\&\leq c^2L^{\frac{2}{3}}\mathbf{E}\bigg(\left[\frac{\Gamma_{cL^{1/3}+1}}{cL^{1/3}}\right]^4\bigg)^{\frac{1}{2}}\mathbf{E}\bigg(\left[\frac{1}{\Gamma_i}\right]^4\bigg)^{\frac{1}{2}}.\end{aligned} \tag{4.88}$$

It remains to point out, on the one hand, that by the law of large number $\mathbf{E}\Big(\Big[\frac{\Gamma_{cL^{1/3}+1}}{cL^{1/3}}\Big]^4\Big)$ converges to 1 as $L \to \infty$ and, on the other hand, that for all $i \in \mathbb{N} \setminus \{0\}$, Γ_i follows a Gamma distribution of parameter $(i, 1)$ which entails that for $i \geq 5$, $\mathbf{E}\Big(\Big[\frac{1}{\Gamma_i}\Big]^4\Big) = \frac{(i-5)!}{(i-1)!}$. Consequently, we can use (4.87) and (4.88) to complete the proof of Step 2.

Proof of Claim 4.9

We use again the random walk representation (3.3), and since $\Gamma_\beta = 1$, we obtain

$$P_{L,\beta}\Big(\frac{N_l}{L^{2/3}} \leq t\Big) = \frac{c_\beta}{\widetilde{Z}_{L,\beta}} \sum_{N=1}^{tL^{2/3}} \mathbf{P}_\beta\left(G_N = L - N, V_{N+1} = 0\right). \tag{4.89}$$

Similarly to what we did in (4.14–4.16), we partition the event $\{G_N = L - N, V_{N+1} = 0\}$ depending on the length r on which the random walk sticks at the origin before its right extremity, that is

$$\begin{aligned} P_{L,\beta}\Big(\frac{N_l}{L^{2/3}} \leq t\Big) &= \sum_{r=0}^{tL^{2/3}-1} \frac{(1 - e^{-\beta/2})}{c_\beta^{r-1} \widetilde{Z}_{L,\beta}} \sum_{N=1}^{tL^{2/3}-r} \mathbf{P}_\beta\left(G_N = L - N - r, V_N \neq 0, V_N V_{N+1} \leq 0\right) \\ &= \sum_{r=0}^{tL^{2/3}-1} \xi_{r,L} \mathbf{P}_\beta\left(\frac{r + \mathfrak{N}_1 + \cdots + \mathfrak{N}_{v_{L-r+1}}}{L^{2/3}} \leq t \,|\, L - r + 1 \in \mathfrak{X}\right), \end{aligned} \tag{4.90}$$

where we recall the definition of $\xi_{r,L}$ in (4.49). This ends the proof of Claim 4.9.

4.4 Identifying the distribution of $\lim_{L\to+\infty} \frac{N_l}{L^{2/3}}$

Let B be a standard Brownian motion on the line; we consider its geometric area and its continuous inverse

$$G_t(B) = \int_0^t |B_s| \, ds\,, \quad g_a = \inf\left\{t > 0 : G_t(B) = a\right\}. \tag{4.91}$$

We aim to identify *formally* the distribution of $\lim_{L\to+\infty} \frac{N_l}{L^{2/3}} =_{law} \sum_{i=1}^{+\infty} Y_i \, \mathfrak{U}_i^{2/3}$ with the distribution of g_1 conditionally on $B_{g_1} = 0$.

Step 1: Identifying the distribution of Y_1

We shall show that Y_1 is distributed as the extension of a Brownian excursion normalized by its area. The Brownian excursion distribution is $\pi_1 = P_{0,0}^{3,1}$ the law of the Bessel bridge of dimension 3 and length 1. We may view this law as the distribution of the excursion conditioned to have length (extension) 1. This is an easy consequence of It's description of the excursion measure (see [30, Theorem 4.2] or [28, section 2]). Indeed let $\zeta(\omega)$ be the extension (length,duration) of an excursion

$$\zeta(\omega) := \inf\left\{t > 0 : \omega(t) = 0\right\}, \tag{4.92}$$

and $\pi_r = P_{0,0}^{3,r}$ be the law of the Bessel bridge of dimension 3 over $[0, r]$. Then under n_+, the positive excursion measure, ζ has density $\frac{1}{2\sqrt{2\pi r^3}}$ and we have

$$n_+(\Gamma) = \int_0^{+\infty} \pi_r(\Gamma) \frac{dr}{2\sqrt{2\pi r^3}} = \int_0^{+\infty} \pi_r(\Gamma)\, n_+(\zeta \in dr)\,. \tag{4.93}$$

The usual scaling operator is $s_c(\omega)(t) = \frac{1}{\sqrt{c}}\omega(ct)$ and we have $\pi_r(F(\omega)) = \pi_1(F(s_{1/r}(\omega)))$. Thus, if $\nu(\omega) = s_{\zeta(\omega)}(\omega)$ is the operator that normalizes the length of the excursion,

$\zeta(\nu(\omega)) = 1$, then we have an independence between the length of an excursion and its shape, easily deduced from (4.93): for any positive measurable F, ψ

$$n_+(F(\nu(\omega))\psi(\zeta(\omega)) = \pi_1(F(\omega))n_+(\psi(\zeta))\,.$$

If we consider now instead of the extension the area

$$A(\omega) = \int_0^{+\infty} \omega(s)\,ds = \int_0^{\zeta(\omega)} \omega(s)\,ds\,, \tag{4.94}$$

then the operator that normalizes the area of an excursion is $\eta(\omega) = s_{A(\omega)^{2/3}}(\omega)$ and we can establish using scaling again that there exists a probability γ_A defined on excursions such that $\gamma_A(A(\omega) = 1) = 1$ and that satisfies for every positive measurable F, ψ:

$$n_+\left[F(\eta(\omega))\psi(A(\omega))\right] = \gamma_A(F(\omega))\eta_+(\psi(A(\omega)))\,. \tag{4.95}$$

It is natural to say that γ_A is the law of the Brownian excursion normalized by its area and it is just a matter of playing with scaling again to show that Y_1 of density proportional to $\frac{1}{x^3}w(\frac{1}{x^{3/2}})$ is distributed as $C_\beta^{2/3}\zeta(\omega)$ under γ_A.

Step 2: A Brownian construction of a $\frac{1}{3}$-stable regenerative set and a formal identity in law.

Observe that if $(\tau_t, t \geq 0)$ is the inverse local time at level 0 of Brownian motion B, then by strong Markov property $(G_{\tau_t}, t \geq 0)$ is a subordinator. Since it has the scaling $(G_{\tau_{ct}}, t \geq 0) =_{law} (c^3 G_{\tau_t}, t \geq 0)$, the closure of its range $\mathcal{R} = \{G_{\tau_t}, t \geq 0\} \cup \left\{G_{\tau_{t-}}, t > 0\right\}$ is a stable $\frac{1}{3}$ regenerative set on $[0, +\infty[$.

Therefore if $(\mathfrak{U}_i)_{i=1}^{+\infty}$ are the interarrivals of $\mathfrak{T} = \mathcal{R} \cap [0, 1]$, we have the representation

$$\{\mathfrak{U}_i, 1 \leq i\} = \left\{G_{\tau_s} - G_{\tau_{s-}} : s > 0, G_{\tau_s} \leq 1\right\}\,. \tag{4.96}$$

Assume that instead $\{\mathfrak{U}_i, 1 \leq i\} = \left\{G_{\tau_s} - G_{\tau_{s-}} : 0 < s \leq t\right\}$. Then the exponential formula for the Poisson process of Brownian excursion yields

$$\mathbb{E}\left[\exp(-\lambda \sum_i \mathfrak{U}_i^{2/3})\right] = \exp\left(-2\int_0^t \left(1 - e^{-\lambda A(w)^{2/3}}\right) n_+(dw)\right)\,. \tag{4.97}$$

Since the Y_i are IID, by considering the marked Poisson process of excursion, we get

$$\mathbb{E}\left[\exp(-\lambda \sum_i Y_i \mathfrak{U}_i^{2/3})\right] = \exp\left(-2\int_0^t \left(1 - \mathbb{E}\left[e^{-\lambda Y_1 A(w)^{2/3}}\right]\right) n_+(dw)\right)\,. \tag{4.98}$$

By the independence of the area of an excursion and its shape (we take $C_\beta = 1$ to simplify notations) (4.95), we obtain

$$\begin{aligned}\mathbb{E}\left[\exp(-\lambda \sum_i Y_i \mathfrak{U}_i^{2/3})\right] &= \exp\left(-2\int_0^t (1 - e^{-\lambda\zeta(w)})n_+(dw)\right)\\ &= \exp\left(-\int_0^t (1 - e^{-\lambda\zeta(w)})n(dw)\right) = \mathbb{E}\left[e^{-\lambda\sum_{s\leq t}(\tau_s - \tau_{s-})}\right] = \mathbb{E}\left[e^{-\lambda\tau_t}\right]\,. \end{aligned}\tag{4.99}$$

We extend formally this identity in law to a sum until the sum of the $\mathfrak{U}_i$ exceeds 1, and we obtain formally

$$\sum_{i=1}^{+\infty} Y_i\,\mathfrak{U}_i^{2/3} =_{law} \sum_{s:G_{\tau_s}\leq 1} \tau_s - \tau_{s-} = g_1 = \inf\left\{t > 0 \int_0^t |B_s|\,ds > 1\right\}\,. \tag{4.100}$$

Step 3: A formal conditioning

We now have to take into account the conditioning. The $(\mathfrak{U}_i)_{i=1}^{+\infty}$ are the interarrivals of $\mathfrak{T} = \mathcal{R} \cap [0,1]$, conditionally on $1 \in \mathfrak{T}$ that is

$$\{\mathfrak{U}_i, 1 \le i\} =_{law} \left\{G_{\tau_s} - G_{\tau_{s-}} : s > 0, G_{\tau_s} \le 1\right\} \quad \text{conditionally on } \{\exists s : G_{\tau_s} = 1\}. \tag{4.101}$$

Since by definition of g we have $\{\exists s : G_{\tau_s} = 1\} = \{B_{g_1} = 0\}$ we can conclude that

$$\sum_{i=1}^{+\infty} Y_i \, \mathfrak{U}_i^{2/3} =_{law} C_\beta^{2/3} g_1 \quad \text{conditionally on } \{B_{g_1} = 0\}\,. \tag{4.102}$$

Let us explain why this conditioning is only formal. The sets on which we condition are of zero probability measure and thus the law of $\mathfrak{T} = \mathcal{R} \cap [0,1]$ conditioned by $1 \in \mathfrak{T}$ is defined in [9] through regular conditional distributions (formulas (1.19) and (1.20)).[2]

5 Fluctuation of the convex envelopes around the Wulff shape: proof of Theorem 2.5

Let us first recall some notations. For each $l \in \mathcal{L}_{N,L}$ we defined in (2.12) the middle line M_l. We also defined in Section 3.1, the T_N transformation that associates with each $l \in \mathcal{L}_{N,L}$ the path $V_l = (T_N)^{-1}(l)$ such that $V_{l,0} = 0$, $V_{l,i} = (-1)^{i-1} l_i$ for all $i \in \{1, \dots, N\}$ and $V_{l,N+1} = 0$. Finally, note that the path M_l can be rewritten with the increments $(U_i)_{i=1}^{N+1}$ of the V_l random walk as

$$M_{l,i} = \sum_{j=1}^{i} (-1)^{j+1} \frac{U_j}{2}, \quad i \in \{1, \dots, N\}. \tag{5.1}$$

In the same spirit, we will need to work with the V random walk sampled from $\mathbf{P}_\beta$ directly. We associate with each V trajectory the process M that is obtained exactly as M_l is obtained from V_l in (5.1), i.e.,

$$M_i = \sum_{j=1}^{i} (-1)^{j+1} \frac{U_j}{2}, \quad i \in \mathbb{N}. \tag{5.2}$$

We recall finally that, for any trajectory $V = (V_i)_{i=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}}$ and any $N \in \mathbb{N}$, $A_N(V)$ is the algebraic area below the V-trajectory up to time N and $G_N(V)$ is its geometric counterpart, i.e., $A_N(V) = \sum_{i=1}^N V_i$ and $G_N(V) = \sum_{i=1}^N |V_i|$.

5.1 Large deviation estimates

In this section, we apply to the probability measure $\mathbf{P}_\beta$ the *exponential tilting* introduced in [14], in order to study $\mathbf{P}_\beta$ conditioned on the large deviation event $\{A_n(V) = qn^2, V_n = 0\}$. Under the tilted probability measure the large deviation event $\{A_n = n^2 q,\, V_n = 0\}$ becomes typical. To that aim, we denote by $\mathfrak{L}(h), h \in \mathbb{R}$ the logarithmic moment generating function of the random walk V, i.e,

$$\mathfrak{L}(h) := \log \mathbf{E}_\beta[e^{hU_1}]. \tag{5.3}$$

[2] There are well known examples of negligible sets A limits of two different sequences A_n and B_n of non negligible sets, and which lead to two different limiting probabilities $\lim \mathbb{P}\,(.\mid A_n) \neq \lim \mathbb{P}\,(.\mid B_n)$ by considering different regular conditional probability measures

From the definition of the law $\mathbf{P}_\beta$ in (1.6), we obviously have $\mathfrak{L}(h) < \infty$ for all $h \in (-\beta/2, \beta/2)$. For the ease of notations, we set $\Lambda_n := (\frac{A_{n-1}}{n}, V_n)$ and we denote its logarithmic moment generating function by $\mathfrak{L}_{\Lambda_n}(H)$ for $H := (h_0, h_1) \in \mathbb{R}^2$, i.e.,

$$\mathfrak{L}_{\Lambda_n}(H) := \log \mathbf{E}_\beta\big[e^{h_0 \frac{A_{n-1}}{n} + h_1 V_n}\big] = \textstyle\sum_{i=1}^n \mathfrak{L}\Big(\big(1 - \frac{i}{n}\big)h_0 + h_1\Big). \tag{5.4}$$

Clearly, $\mathfrak{L}_{\Lambda_n}(H)$ is finite for all $H \in \mathcal{D}_n$ with

$$\mathcal{D}_n := \Big\{(h_0, h_1) \in \mathbb{R}^2 \colon h_1 \in \big(-\tfrac{\beta}{2}, \tfrac{\beta}{2}\big),\ (1 - \tfrac{1}{n})h_0 + h_1 \in \big(-\tfrac{\beta}{2}, \tfrac{\beta}{2}\big)\Big\}. \tag{5.5}$$

We also introduce $\mathfrak{L}_\Lambda$ the continuous counterpart of $\mathfrak{L}_{\Lambda_n}$ as

$$\mathfrak{L}_\Lambda(H) := \textstyle\int_0^1 \mathfrak{L}(x h_0 + h_1) dx, \tag{5.6}$$

which is defined on

$$\mathcal{D} := \Big\{(h_0, h_1) \in \mathbb{R}^2 \colon h_1 \in \big(-\tfrac{\beta}{2}, \tfrac{\beta}{2}\big),\ h_0 + h_1 \in \big(-\tfrac{\beta}{2}, \tfrac{\beta}{2}\big)\Big\}. \tag{5.7}$$

With the help of (5.4) and for $H = (h_0, h_1) \in \mathcal{D}_n$, we define the H-tilted distribution by

$$\frac{\mathrm{d}\mathbf{P}_{n,H}}{\mathrm{d}\mathbf{P}_\beta}(V) = e^{h_0 \frac{A_{n-1}}{n} + h_1 V_n - \mathfrak{L}_{\Lambda_n}(H)}. \tag{5.8}$$

For a given $n \in \mathbb{N}$ and $q \in \frac{\mathbb{N}}{n}$, the exponential tilt is given by $H_n^q := (h_{n,0}^q, h_{n,1}^q)$ which, by Lemma 5.5 in Section 5.1 of [10], is the unique solution of

$$\mathbf{E}_{n,H}\big(\tfrac{\Lambda_n}{n}\big) = \nabla\big[\tfrac{1}{n}\mathfrak{L}_{\Lambda_n}\big](H) = (q, 0). \tag{5.9}$$

Then, we define the continuous counterpart of H_n^q by $\widetilde{H}(q,0) := (\widetilde{h}_0(q,0), \widetilde{h}_1(q,0))$ which is the unique solution of the equation

$$\nabla \mathfrak{L}_\Lambda(H) = (q, 0). \tag{5.10}$$

In the proofs of Theorem 5.10 and Propoosition 5.3, we will use the following Proposition which is a direct consequence of [10, Proposition 2.3].

Proposition 5.1. *For $[q_1, q_2] \subset (0, \infty)$, there exists $\mathcal{K}$ a compact subset of $\mathcal{D}$ and $n_0 \in \mathbb{N}$ such that for every $q \in [q_1, q_2]$*

$$\begin{aligned} H_n^q &\in \mathcal{K}, \quad \forall n \geq n_0, \\ \widetilde{H}(q, 0) &\in \mathcal{K}. \end{aligned} \tag{5.11}$$

5.2 Outline of the proof

We recall the definition of $\widetilde{P}_{L,\beta}$ in (2.8). As stated in Remark 2.7, the proof of Theorem 2.5 will be complete once we show the convergence in law (2.14). To that aim we will prove Proposition 5.2 and 5.3 below, that is a finite dimensional convergence and a tension argument which will be sufficient to prove (2.14).

Given $\bar{t} = (t_1, \ldots, t_{r_1})$ with $0 < t_1 < \cdots < t_{r_1} < 1$, for $\bar{x} \in \mathbb{R}^{r_1}$ we let $g_{H,\bar{t}}(\bar{x})$ be the density of the Gaussian vector $\xi_H(\bar{t}) = (\xi_H(t_1), \ldots, \xi_H(t_{r_1}))$ and let $f_{H,\bar{t}}(z_0, z_1, \bar{x})$ be the density of the Gaussian vector $(\int_0^1 \xi_H(s)\, ds, \xi_H(1), \xi_H(t_1), \ldots, \xi_H(t_{r_1}))$. Finally let

$$f^c_{H,\bar{t}}(\bar{y}) = \frac{f_{H,\bar{t}}(0,0,\bar{y})}{\int f_{H,\bar{t}}(0,0,\bar{x})\, d\bar{x}}$$

be the density of the law of $\xi_H(\bar{t})$ conditional on $\int_0^1 \xi_H(s)\, ds = 0 = \xi_H(1)$.

Proposition 5.2. *For $\beta > \beta_c$ and $(r_1, r_2) \in \mathbb{N}^2$, consider $\bar{s} = (s_i)_{i=1}^{r_1}$ and $\bar{t} = (t_i)_{i=1}^{r_2}$ two ordered sequences in $[0,1]$. Set $m_L = a(\beta)\sqrt{L}$. We have that*

$$\lim_{L\to\infty} \sup_{(\bar{x},\bar{y})\in\mathbb{Z}^{r_1}\times\mathbb{N}^{r_2}} \Big|(m_L)^{\frac{r_1+r_2}{2}} \widetilde{P}_{L,\beta}\big[H_{\bar{s},\bar{t}}(\bar{x},\bar{y})\big] - g_{\beta,\bar{s}}\Big(\tfrac{\bar{x}}{\sqrt{m_L}}\Big)\, f_{\beta,\bar{t}}\Big(\tfrac{\bar{y}}{\sqrt{m_L}}, m_L\gamma_\beta^*\Big)\Big| = 0, \tag{5.12}$$

with

$$H_{\bar{s},\bar{t}}(\bar{x},\bar{y}) = \left\{N_l\left(2\widetilde{M}_l(\bar{s}), |\widetilde{V}_l(\bar{t})|\right) = (\bar{x},\bar{y})\right\}, \tag{5.13}$$

$$g_{\beta,\bar{s}}(\bar{x}) = g_{\widetilde{H}(q_\beta,0),\bar{s}}(\bar{x}) \quad \textit{and} \quad f_{\beta,\bar{t}}(\bar{y},\varphi) = \frac{1}{2}(f^c_{\widetilde{H}(q_\beta,0),\bar{t}}(\bar{y}-\varphi(\bar{t})) + f^c_{\widetilde{H}(q_\beta,0),\bar{t}}(\bar{y}+\varphi(\bar{t}))). \tag{5.14}$$

Proposition 5.3. *For $\beta > \beta_c$, the sequence of probability laws $(\widehat{Q}_{L,\beta})_{L\in\mathbb{N}}$ is tight.*

Let us give here the key idea behind the proofs of Propositions 5.2 and 5.3. We will first prove the counterpart of Propositions 5.2 and 5.3 with the processes $\widetilde{M}$ and $\widetilde{V}$ sampled from $\mathbf{P}_\beta$ (cf. 5.2) conditional on $V_N = 0, A_N(V) = qN^2$ ($q > 0$). The reason for these two intermediate results is that they can be obtained with the tilting of $\mathbf{P}_\beta$ exposed in Section 5.1 and first introduced in [14]. Then, we will translate these two results in terms of the two processes $\widetilde{M}_l$ and $\widetilde{V}_l$ (see (5.1)) obtained with l sampled from $\widetilde{P}_{L,\beta}$. However, this last step is difficult because the conditioning that emerges from the T_N transformation (see section (3.1)) involves the geometric area below the V_l random walk rather than its algebraic counterpart. This is the reason why we state in Section 5.3 a couple of preparatory Lemmas indicating that, when the algebraic area below V is abnormally large (qN^2 instead of the typical $N^{3/2}$) then the geometric area below V is not only also abnormally large but is fairly close to the algebraic area. These Lemmas will be proven in Section 5.7, except for Lemma 5.6 that was already proven in [10].

In Section 5.4 we state and prove a local limit theorem for any finite dimensional joint distribution of the middle line M and the profile $|V|$ with V sampled from $\mathbf{P}_\beta(\cdot|V_N = 0, |A_N(V)| = qN^2)$ as $N \to \infty$. As a by product of this local limit theorem we will observe that asymptotically the rescaled profile $|\widetilde{V}|$ and the rescaled middle line $\widetilde{M}$ decorrelate. Section 5.4 can be considered as the first part of the proof of Proposition 5.2. We will indeed prove in Section 5.5 that the latter asymptotic decorrelation still holds true with $|\widetilde{V}_l|$ and $\widetilde{M}_l$ when l is sampled from $\widetilde{P}_{L,\beta}$ and $\beta > \beta_c$. This will complete the proof of Proposition 5.2. Similarly, Section 5.6 can be seen as the first part of the proof of Proposition 5.3 since we prove the tightness of $\widetilde{M}$ and $\widetilde{V}$ under $\mathbf{P}_\beta(\cdot|V_N = 0, A_N(V) = qN^2)$ as $N \to \infty$. However, we will not display the part of the proof showing that this tightness is still satisfied under $\widetilde{P}_{L,\beta}$ since this can be done by mimicking the proof in Section 5.5.

5.3 Preparations

Lemma 5.4 shows that the probability, under the polymer measure, that the rescaled horizontal excursion deviates from $a(\beta)$ by more than a given vanishing quantity decays faster than any given polynomial provided the vanishing quantity decreases slowly enough.

Lemma 5.4. *We set $\eta_L = L^{-1/8}$. For $\beta > \beta_c$ and $\alpha > 0$ we have*

$$\lim_{L\to\infty} L^\alpha P_{L,\beta}\big[\big(B_{\eta_L,L}\big)^c\big] = 0\,, \tag{5.15}$$

with

$$B_{\eta,L} = \Big\{\tfrac{N_l}{\sqrt{L}} \in a(\beta) + [-\eta,\eta]\Big\}. \tag{5.16}$$

Lemma 5.5 indicates that, provided we choose a constant $c > 0$ large enough, the probability that the algebraic area and the geometric area described by V_l differ from each other by more than $c(\log L)^4$ tends to 0 faster than any polynomial.

Lemma 5.5. *For $\beta > \beta_c$ and $\alpha > 0$ there exists a $c > 0$ such that*

$$\lim_{L\to\infty} L^{\alpha}\, P_{L,\beta}\big(|A_{N_l}(V_l)| \notin [L - N_l - c(\log L)^4, L - N_l]\big) = 0. \tag{5.17}$$

Lemma 5.6 was proven in [10, proposition 2.4]. It identifies the sub-exponential decay rate of the event $\{V_N = 0, A_N(V) = qN^2\}$ when V is sampled from $\mathbf{P}_\beta$.

Lemma 5.6. [Proposition 2.4 in [10]] *For $[q_1, q_2] \subset (0, \infty)$, there exist $C_1 > C_2 > 0$ and $n_0 \in \mathbb{N}$ such that for all $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$ we have*

$$\frac{C_2}{N^2} e^{-\rho_\beta(q)N} \le \mathbf{P}_\beta(A_N(V) = qN^2, V_N = 0) \le \frac{C_1}{N^2} e^{-\rho_\beta(q)N}\,, \tag{5.18}$$

where $\rho_\beta(q) := \mathfrak{L}_\Lambda(\widetilde{H}(q,0)) - q\widetilde{h}_0(q,0)$, so that ρ_β is $\mathcal{C}^\infty$ and we have $\widetilde{G}(a) = a(\log\Gamma_\beta - \rho_\beta(\frac{1}{a^2}))$ (recall (5.87)).

Lemma 5.7 ensures that, when V is sampled from $\mathbf{P}_\beta$ conditional on $V_N = 0, |A_N(V)| = qN^2$, the probability that the geometric area described by V differs from the algebraic area by more than $(\log N)^4$ tends to 0 faster than any polynomial.

Lemma 5.7. *For any $[q_1, q_2] \subset (0, \infty)$ and $\alpha, c > 0$ we have*

$$\lim_{N\to\infty} \sup_{q\in[q_1,q_2]} N^{\alpha}\, \mathbf{P}_\beta\big(G_N(V) \ge qN^2 + c(\log N)^4 \,|\, V_N = 0, |A_N(V)| = qN^2\big) = 0. \tag{5.19}$$

We recall that all Lemmas of this Section are proven in Section 5.7.

5.4 Asymptotic decorrelation of the middle line and of the profile

In this section, we prove the following Lemma that gives us a local limit theorem for the paths $V = (V_i)_{i=0}^N$ and $M = (M_i)_{i=0}^N$ simultaneously when V is sampled from $\mathbf{P}_\beta(\cdot \,|\, V_N = 0, A_N(V) = qN^2)$. This local limit theorem is reinforced by the fact that it is uniform in q belonging to any compact set of $(0, \infty)$.

Lemma 5.8. *For $[q_1, q_2] \subset (0, \infty)$ and $(r_1, r_2) \in \mathbb{N}^2$, we have*

$$\lim_{N\to+\infty} \sup_{q\in[q_1,q_2]} \sup_{(\bar{x},\bar{y})\in\mathbb{Z}^{r_1+r_2}} \Big| N^{\frac{r_1+r_2}{2}}\, \mathbf{P}_\beta(H_{\bar{s},\bar{t}}(\bar{x},\bar{y}) \,|\, W_{N,L,qN^2}) - \hat{f}_{\widetilde{H}(q,0),\bar{t}}\left(\frac{\bar{y}}{N^{1/2}}, N^{1/2}\gamma_q^*\right) g_{\widetilde{H}(q,0),\bar{s}}\left(\frac{\bar{x}}{N^{1/2}}\right)\Big| = 0\,, \tag{5.20}$$

with

$$W_{N,L,b} = \big\{\, |A_N(V)| = b,\, V_{N+1} = 0,\, G_N(V) \in \{b, \dots, b + c(\log L)^4\}\big\},$$

and

$$\hat{f}_{H,\bar{t}}(\bar{y}, \varphi) = \frac{1}{2}\left(f^c_{H,\bar{t}}(\bar{y} - \varphi(\bar{t})) + f^c_{H,\bar{t}}(\bar{y} + \varphi(\bar{t}))\right).$$

Remark 5.9. For a given $\bar{s}$ and $\bar{t}$, and $[q_1, q_2] \subset (0, \infty)$, there exists a $M > 0$ such that

$$\sup_{q\in[q_1,q_2]} \sup_{(\bar{x},\bar{y})\in\mathbb{R}^{r_1+r_2}} f^c_{\widetilde{H}(q,0),\bar{t}}(\bar{y})\, g_{\widetilde{H}(q,0),\bar{s}}(\bar{x}) \le M. \tag{5.21}$$

Proof of Lemma 5.8

First of all we note that, thanks to Lemma 5.7, it is sufficient to prove Lemma 5.8 with the conditioning $\{V_N = 0, |A_N(V)| = qN^2\}$ instead of W_{N,L,qN^2}. We will first prove Lemma 5.8 subject to Theorem 5.10 that is stated below and then, we will prove Theorem 5.10.

Theorem 5.10. *For any* $[q_1, q_2] \subset \mathbb{R}$ *we have*

$$\lim_{N\to+\infty} \sup_{q\in[q_1,q_2]} \sup_{(z_0,z_1,\bar{x},\bar{y})\in\mathbb{R}^{2+r_1+r_2}} \Big| N^{2+\frac{r_1+r_2}{2}} P_{N,H_N^q}\left(A_N = N^2 q + z_0, V_N = z_1, V_{\lfloor N\bar{t}\rfloor} = \bar{y}, 2M_{\lfloor N\bar{s}\rfloor} = \bar{x}\right) - f_{\widetilde{H}(q,0),\bar{t}}\left(\frac{z_0}{N^{3/2}}, \frac{z_1}{N^{1/2}}, \frac{\bar{y} - N\gamma_q^*(\bar{t})}{N^{1/2}}\right) g_{\widetilde{H}(q,0),\bar{t}}\left(\frac{\bar{x}}{N^{1/2}}\right)\Big| = 0\,. \tag{5.22}$$

Let $k_H(z_0, z_1)$ be the density of the law of $(\int_0^1 \xi_H(s)\, ds, \xi_H(1))$. Then we have $k_H(z_0, z_1) = \int f_{H,\bar{t}}(z_0, z_1, \bar{y})\, d\bar{y}$ and we have the limit theorem (Proposition 6.1 of [10])

$$\lim_{N\to\infty} N^2 P_{N,H_N^q}\left(V_N = 0, A_N = qN^2\right) = k_{\widetilde{H}(q,0)}(0,0)\,. \tag{5.23}$$

Therefore, since we can write

$$\begin{aligned} &N^{\frac{r_1+r_2}{2}} \mathbf{P}_\beta(H_{\bar{s},\bar{t}}(\bar{x},\bar{y}) \,|\, V_N = 0, |A_N(V)| = qN^2) = \\ &\quad \frac{N^{2+\frac{r_1+r_2}{2}} P_{N,H_N^q}\left(A_N = N^2 q, V_N = 0, V_{\lfloor N\bar{t}\rfloor} = \bar{y}, 2M_{\lfloor N\bar{s}\rfloor} = \bar{x}\right)}{N^2\Big(P_{N,H_N^q}(V_N = 0, A_N = qN^2) + P_{N,H_N^q}(V_N = 0, A_N = -qN^2)\Big)} \\ &\quad + \frac{N^{2+\frac{r_1+r_2}{2}} P_{N,H_N^q}\left(A_N = -N^2 q+, V_N = 0, V_{\lfloor N\bar{t}\rfloor} = \bar{y}, 2M_{\lfloor N\bar{s}\rfloor} = \bar{x}\right)}{N^2\Big(P_{N,H_N^q}(V_N = 0, A_N = qN^2) + P_{N,H_N^q}(V_N = 0, A_N = -qN^2)\Big)}\,. \end{aligned} \tag{5.24}$$

We now can use some symmetry argument (the symmetry of the distribution of the increments of the geometric random walk) to say that

$$\begin{aligned} P_{N,H_N^q}\left(V_N = 0, A_N = qN^2\right) &= P_{N,H_N^q}\left(V_N = 0, A_N = -qN^2\right) \\ \gamma_{-q}^*(t) = -\gamma_q^*(t)\,, \quad \widetilde{H}(-q,0) = -\widetilde{H}(q,0), &\quad f^c_{H,\bar{t}}(\bar{y}) = f^c_{-H,\bar{t}}(\bar{y}) = f^c_{H,\bar{t}}(-\bar{y})\,. \end{aligned} \tag{5.25}$$

Therefore we obtain the desired result by applying Theorem 5.10 and combining it with the definition of $f^c_{H,t}$.

Proof of Theorem 5.10

Let us first quickly rephrase the statement of Theorem 5.10 in a simplified context to ease its comprehension. To that aim, we consider a random walk $S_0 = 0, S_n = X_1 + \cdots + X_n$ with IID increment and we build an alternating sign random walk $\bar{S}_n := \sum_{i=1}^n (-1)^{i+1} X_i$. If X_1 is square integrable and, say, $\mathbb{E}[X_1] = 0$, $\mathbb{E}\left[X_1^2\right] = 1$, then by the Central Limit Theorem, $\frac{S_n}{\sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0,1)$. Furthermore, we have asymptotic decorrelation, i.e.,

$$\frac{1}{\sqrt{n}}(S_n, \bar{S}_n) \xrightarrow{d} (Z, \bar{Z}),$$

where $(Z, \bar{Z})$ is a pair of independent $\mathcal{N}(0,1)$ distributed random variables, and this convergence can be lifted to the level of processes, i.e.,

$$\frac{1}{\sqrt{n}}(S_{\lfloor nt\rfloor}, \bar{S}_{\lfloor nt\rfloor}, ; t \ge 1) \xrightarrow{d} (B_t, \bar{B}_t, t \ge 1),$$

with $(B, \bar{B})$ a pair of independent standard Brownian motions. Theorem 5.10 asserts that such decorrelation can be extended to properly conditioned processes, in the sense of finite distributions.

The proof of Theorem 5.10 is quite involved but very similar to the proof given in [15]. The main difference comes from fact that we consider simultaneously the V process, on which the conditioning acts, and the middle line M, while in [15] only V is considered. For this reason, in our proof below, we insist mostly on those arguments that differ from [15].

Recall that $V_{\lfloor N\bar{t}\rfloor} = (V_{\lfloor Nt_1\rfloor}, \dots, V_{\lfloor Nt_{r_1}\rfloor})$ and for $\bar{s} \in (0,1)^{r_2}$, $M_{\lfloor N\bar{s}\rfloor} = (M_{\lfloor Ns_1\rfloor}, \dots, M_{\lfloor Ns_{r_2}\rfloor})$. The proof is a copy of the classic proof of the local central limit theorem. We set

$$X_N := \left(\frac{z_0}{N^{3/2}}, \frac{z_1}{N^{1/2}}, \frac{\bar{y} - E_{N,H_N^q}\left(V_{\lfloor N\bar{t}\rfloor}\right)}{N^{1/2}}, \frac{\bar{x} - 2E_{N,H_N^q}\left(M_{\lfloor N\bar{s}\rfloor}\right)}{N^{1/2}} \right). \tag{5.26}$$

Recall that by construction of H_N^q, we have $E_{N,H_N^q}(A_N) = N^2 q$. Hence, by Fourier inversion formula

$$N^{2+\frac{r_1+r_2}{2}} P_{N,H_N^q}\left(A_N = N^2 q + z_0, V_N = z_1, V_{\lfloor N\bar{t}\rfloor} = \bar{y}, M_{\lfloor N\bar{s}\rfloor} = \bar{x}\right) = (2\pi)^{-(2+r_1+r_2)} \int_{\mathcal{A}_N} e^{-i<T,X_N>} \hat{\varphi}_{N,H_N^q}(T)\, dT, \tag{5.27}$$

with $T = (\tau_0, \tau_1, \bar{\kappa}, \bar{\eta})$, $\mathcal{A}_N$ the domain

$$\mathcal{A}_N := \left\{ T : |\tau_0| \le N^{3/2}\pi, |\tau_1| \le \sqrt{N}\pi, |\kappa_i| \le \sqrt{N}\pi, |\eta_j| \le \sqrt{N}\pi \right\}, \tag{5.28}$$

and $\hat{\varphi}_{N,H_N^q}(T)$ the characteristic function

$$\hat{\varphi}_{N,H_N^q}(T) := E_{N,H_N^q}\left[e^{i\left(\frac{\tau_0}{N^{3/2}}(A_N - N^2 q) + \frac{1}{N^{1/2}}(\tau_1 V_N + <\bar{\kappa}, V_{\lfloor N\bar{t}\rfloor} - E_{N,H_N^q}\left(V_{\lfloor N\bar{t}\rfloor}\right)> + <\bar{\eta}, 2M_{\lfloor N\bar{s}\rfloor} - E_{N,H_N^q}\left(2M_{\lfloor N\bar{s}\rfloor}\right)>)\right)} \right].$$

Therefore

$$\begin{aligned} R_N &:= N^{2+\frac{r_1+r_2}{2}} P_{N,H_N^q}\left(A_N = N^2 q + z_0, V_N = z_1, V_{\lfloor N\bar{t}\rfloor} = \bar{y}, M_{\lfloor N\bar{s}\rfloor} = \bar{x}\right) \\ &\quad - f_{\widetilde{H}(q,0),\bar{t}}\left(\frac{z_0}{N^{3/2}}, \frac{z_1}{N^{1/2}}, \frac{\bar{y} - E_{N,H_N^q}\left(V_{\lfloor N\bar{t}\rfloor}\right)}{N^{1/2}} \right) g_{\widetilde{H}(q,0),\bar{s}}\left(\frac{\bar{x} - E_{N,H_N^q}\left(2M_{\lfloor N\bar{s}\rfloor}\right)}{N^{1/2}} \right) = \\ &\quad C \int_{\mathcal{A}_N} e^{-i<T,X_N>} \left(\hat{\varphi}_{N,H_N^q}(T) - \bar{\varphi}_{\widetilde{H}(q,0)}(T) \right) dT, \end{aligned} \tag{5.29}$$

with $\bar{\varphi}_{\widetilde{H}(q,0)}(T)$ the characteristic function of the Gaussian vector with density $f_{H,\bar{t}}(z_0, z_1, \bar{y}) g_{H,\bar{s}}(\bar{x})$ that is the Gaussian vector $(\int_0^1 \xi_H(s)\, ds, \xi_H(1), \xi_H(\bar{t}), \bar{\xi}_H(\bar{s}))$ where $\bar{\xi}_H$ is an independent copy of ξ_H.

Hence,

$$|R_N| \le C \int_{\mathcal{A}_N} \left| \hat{\varphi}_{N,H_N^q}(T) - \bar{\varphi}_{\widetilde{H}(q,0)}(T) \right| dT = \sum_{i=1}^{4} J_i^{(q)}, \tag{5.30}$$

with

$$\begin{aligned} J_1^{(q)} &= \int_{\Gamma_1} \left| \hat{\varphi}_{N,H_N^q}(T) - \bar{\varphi}_{\widetilde{H}(q,0)}(T) \right| dT, & \Gamma_1 &= [-A, A]^{2+r_1+r_2}, \\ J_2^{(q)} &= \left| \int_{\Gamma_2} \bar{\varphi}_{\widetilde{H}(q,0)}(T)\, dT \right|, & \Gamma_2 &= \mathbb{R}^{2+r_1+r_2} \setminus \Gamma_1, \\ J_3^{(q)} &= \int_{\Gamma_3} \left| \hat{\varphi}_{N,H_N^q}(T) \right| dT, & \Gamma_3 &= \left\{ T : |t_i| \le \Delta\sqrt{N} \right\} \setminus \Gamma_1, \\ J_4^{(q)} &= \int_{\Gamma_4} \left| \hat{\varphi}_{N,H_N^q}(T) \right| dT, & \Gamma_4 &= \mathcal{A}_N \setminus (\Gamma_1 \cup \Gamma_3), \end{aligned} \tag{5.31}$$

where $A, \Delta > 0$ are positive constants. We can bound $J_i^{(q)}$ for $i = 2,3,4$ exactly with the same procedure used in [10], Proposition 6.1. For this reason, we shall focus on proving that

$$\forall A > 0\,, \quad \lim_{N\to+\infty} \sup_{q\in[q_1,q_2]} J_1^{(q)} = 0\,.$$

It is enough to prove that

$$\lim_{N\to+\infty} \sup_{q\in[q_1,q_2]} \sup_{T\in\Gamma_1} \left|\hat{\varphi}_{N,H_N^q}(T) - \bar{\varphi}_{\widetilde{H}(q,0)}(T)\right| = 0\,. \tag{5.32}$$

To this end we shall note $\bar{\Lambda}_N = (\frac{A_N}{N}, V_N, V_{\lfloor N\bar{t}\rfloor}, 2M_{\lfloor N\bar{s}\rfloor})$ and consider the moment generating function

$$\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H,\bar{\kappa},\bar{\eta}) := \log \mathbf{E}_\beta\left[\exp\left(h_0\frac{A_N}{N} + h_1 V_N + < \bar{\kappa}, V_{\lfloor N\bar{t}\rfloor} > + < \bar{\eta}, M_{\lfloor N\bar{s}\rfloor} >\right)\right]. \tag{5.33}$$

Observe that

$$\begin{aligned}\log\hat{\varphi}_{N,H_N^q}(T) = {}& \bar{\mathfrak{L}}_{\bar{\Lambda}_N}\left(H_N^q + \frac{i}{N^{1/2}}(\tau_0,\tau_1), \frac{i}{N^{1/2}}\bar{\kappa}, \frac{i}{N^{1/2}}\bar{\eta}\right)\\ &- \bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0) - \frac{i}{N^{1/2}} < T, E_{N,H_N^q}\left(\bar{\Lambda}_N\right) > .\end{aligned}$$

Therefore, an order 2 Taylor expansion gives

$$\log\hat{\varphi}_{N,H_N^q}(T) = -\frac{1}{2} < \frac{1}{N}\,\mathrm{Hess}\,\bar{\mathfrak{L}}_{\bar{\Lambda}_n}(H_N^q,0,0)T, T > +\alpha_N, \tag{5.34}$$

with $\sup_N \sup_{q\in[q_1,q_2]} N^{1/2}\alpha_N < +\infty$.

We can write the moment generating function explicitly as

$$\bar{\mathfrak{L}}_{\bar{\Lambda}_n}(H,\bar{\kappa},\bar{\eta}) = \sum_{1\le i\le N} L\left(\left(1-\frac{i}{N}\right)h_0 + h_1 + \sum_{k:\lfloor Nt_k\rfloor\ge i} t_k + \sum_{m:\lfloor Ns_m\rfloor \ge i}(-1)^{i+1}\eta_m\right). \tag{5.35}$$

Therefore, thanks to Proposition 2.3 and Lemma 5.1 of [10], we have for $i,j\in\{0,1\}$

$$\frac{1}{N}\partial^2_{h_ih_j}\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0) = \frac{1}{N}\partial^2_{h_ih_j}\mathfrak{L}_{\Lambda_N}(H_N^q) \to \partial^2_{h_ih_j}\mathfrak{L}_{\Lambda}(\widetilde{H}(q,0)). \tag{5.36}$$

Furthermore,

$$\frac{1}{N}\partial^2_{\kappa_l^2}\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0) = \frac{1}{N}\sum_{i=1}^{\lfloor Nt_l\rfloor}\mathfrak{L}''\left(\left(1-\frac{i}{N}\right)h_0^{N,q} + h_1^{N,q}\right) \to \int_0^{t_l}\mathfrak{L}''((1-x)\widetilde{h}_0, +\widetilde{h}_1)\,dx\,. \tag{5.37}$$

If $l < m$ then

$$\frac{1}{N}\partial^2_{\kappa_l\kappa_m}\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0) = \frac{1}{N}\partial^2_{\kappa_l^2}\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0), \tag{5.38}$$

and converges to the preceding limit. We have similarly,

$$\frac{1}{N}\partial^2_{\eta_m^2}\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0) = \frac{1}{N}\sum_{i=1}^{\lfloor Ns_m\rfloor}\mathfrak{L}''\left(\left(1-\frac{i}{N}\right)h_0^{N,q} + h_1^{N,q}\right) \to \int_0^{s_m}\mathfrak{L}''((1-x)\widetilde{h}_0, +\widetilde{h}_1)\,dx\,. \tag{5.39}$$

There remains to understand the cross term; we have:

$$\frac{1}{N}\partial^2_{\kappa_l\eta_m}\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0) = \frac{1}{N}\sum_{i=1}^{\lfloor Nt_l\rfloor\wedge\lfloor Ns_m\rfloor}(-1)^{i+1}\mathfrak{L}''\left(\left(1-\frac{i}{N}\right)h_0^{N,q} + h_1^{N,q}\right) \to 0\,, \tag{5.40}$$

since by regrouping the alternating signs two by two, we have

$$\left|\mathfrak{L}''\Big(\Big(1-\frac{i}{N}\Big)h_0^{N,q}+h_1^{N,q}\Big)-\mathfrak{L}''\Big(\Big(1-\frac{i-1}{N}\Big)h_0^{N,q}+h_1^{N,q}\Big)\right|\le\frac{C}{N}\,, \tag{5.41}$$

(thanks to the boundedness of the third derivative $\mathfrak{L}'''(z)$ on a compact of $(-\beta/2,\beta/2)$).

Therefore, as a whole, the Hessian matrix $\frac{1}{N}\,\mathrm{Hess}\,\bar{\mathfrak{L}}_{\bar{\Lambda}_N}(H_N^q,0,0)$ converges uniformly on $[q_1,q_2]$ to the covariance matrix of the vector $(\int_0^1\xi_H(s)\,ds,\xi_H(1),\xi_H(\bar{t}),\bar{\xi}_H(\bar{s}))$ and this establishes the convergence $\lim_{N\to+\infty}R_N=0$.

To conclude, we need to prove now that in the expression of R_N, we can replace the quantities $E_{N,H_N^q}\left(V_{\lfloor N\bar{t}\rfloor}\right)$ and $E_{N,H_N^q}\left(M_{\lfloor N\bar{s}\rfloor}\right)$ by respectively $N\gamma_q^*(\bar{t})$ and 0.

We first observe that when $H=(h_0,h_1)$ lives in a compact set of $\mathcal{D}$ (recall (5.7)), which is the case if $q\in[q_1,q_2]$ and $H=H_N^q$ or $H=\widetilde{H}(q,0)$ (recall Proposition 5.1), then the variances of $\xi_H(t)$ have a positive lower bound, and therefore there exists a uniform Lipschitz constant C such that for any $\bar{x},\bar{x}'\in\mathbb{R}^{r_1}$, $\bar{y},\bar{y}'\in\mathbb{R}^{r_2}$, z_0,z_1,z_0',z_1' and any $q\in[q_1,q_2]$

$$\left|f_{\widetilde{H}(q,0),\bar{t}}(z_0,z_1,\bar{y})-f_{\widetilde{H}(q,0),\bar{t}}(z_0',z_1',\bar{y}')\right|\le C(\|\bar{y}-\bar{y}'\|+\|z-z'\|)\,, \tag{5.42}$$

$$\left|g_{\widetilde{H}(q,0),\bar{s}}(\bar{x})-g_{\widetilde{H}(q,0),\bar{s}}(\bar{x}')\right|\le C\,\|\bar{x}-\bar{x}'\|\,. \tag{5.43}$$

The second observation is that with $h_{N,i}=(1-\frac{i}{N})h_0^{N,q}+h_1^{N,q}$ we have

$$\frac{1}{N}E_{N,H_N^q}\left(V_{\lfloor N\bar{t}\rfloor}\right)=\frac{1}{N}\sum_{i=1}^{\lfloor Nt\rfloor}\mathfrak{L}'(h_{N,i})\xrightarrow{N\to+\infty}\int_0^t\mathfrak{L}'((1-x)\widetilde{h}_0(q,0)+\widetilde{h}_1(q,0))\,dx=\gamma_q^*(t)\,, \tag{5.44}$$

and that, thanks to the boundedness on compact sets of $\mathfrak{L}''$, this convergence holds uniformly.

Similarly

$$2\frac{1}{N}E_{N,H_N^q}\left(M_{\lfloor N\bar{s}\rfloor}\right)=\frac{1}{N}\sum_{i=1}^{\lfloor Nt\rfloor}(-1)^{i+1}\mathfrak{L}'(h_{N,i})\to 0\,, \tag{5.45}$$

and this convergence holds uniformly, since by grouping the alternating signs two by two, we have

$$|\mathfrak{L}'(h_{N,i})-\mathfrak{L}'(h_{N,i+1})|\le\frac{C}{N}\,. \tag{5.46}$$

The proof of Theorem 5.10 is therefore complete.

5.5 Proof of Proposition 5.2

Our aim is to prove the local limit Theorem stated in Proposition 5.2. To that aim, we need to consider two sequences of functions, i.e., $\psi:=(\psi_L)_{L\in\mathbb{N}}$ and $\widetilde{\psi}:=(\widetilde{\psi}_L)_{L\in\mathbb{N}}$, such that for every $L\in\mathbb{N}$, $\psi_L,\widetilde{\psi}_L\colon\ \mathbb{Z}^{r_1+r_2}\to[0,\infty)$. Those function sequences are said to be equivalent and it is denoted by $\psi\sim\widetilde{\psi}$ if

$$\lim_{L\to\infty}\sup_{(\bar{x},\bar{y})\in\mathbb{Z}^{r_1+r_2}}L^{\frac{r_1+r_2}{4}}\left|\psi_L(\bar{x},\bar{y})-\widetilde{\psi}_L(\bar{x},\bar{y})\right|=0. \tag{5.47}$$

We set

$$\begin{aligned}&\psi_{1,L}(\bar{x},\bar{y})=\widetilde{P}_{L,\beta}\big[H_{\bar{s},\bar{t}}(\bar{x},\bar{y})\big],\\&\psi_{5,L}(\bar{x},\bar{y})=m_L^{-\frac{r_1+r_2}{2}}g_{\beta,\bar{s}}\Big(\tfrac{\bar{x}}{\sqrt{m_L}}\Big)\,f_{\beta,\bar{t}}\Big(\tfrac{\bar{y}}{\sqrt{m_L}},\sqrt{m_L}\gamma_\beta^*\Big),\end{aligned} \tag{5.48}$$

and the proof of Theorem 5.2 will be complete once we show that $\psi_1 \sim \psi_5$. To achieve this equivalence we introduce 3 intermediate function sequences ψ_2, ψ_3 and ψ_4 and we divide the proof into 4 steps. For $i \in \{1,2,3,4\}$, the i-th step consists in proving that $\psi_i \sim \psi_{i+1}$ so that at the end of the fourth step we can state that $\psi_1 \sim \psi_5$.

In steps 1 and 2, we will use the fact that for $i \in \{1,2,3\}$ and $L \in \mathbb{N}$, the $\psi_{i,L}$ functions are of the form $\psi_{i,L} = \frac{A_{i,L}}{B_{i,L}}$ such that an equivalence of type (5.47) between ψ_j and ψ_k will be proven once we show that

$$\begin{aligned} &(i) \quad \lim_{L\to\infty} \sup_{(\bar{x},\bar{y})\in\mathbb{Z}^{r_1+r_2}} L^{\frac{r_1+r_2}{4}} \frac{|A_{j,L}-A_{k,L}|}{B_{j,L}} = 0, \\ &(ii) \quad \lim_{L\to\infty} \sup_{(\bar{x},\bar{y})\in\mathbb{Z}^{r_1+r_2}} L^{\frac{r_1+r_2}{4}} \frac{|A_{k,L}|}{B_{j,L}} \frac{|B_{j,L}-B_{k,L}|}{B_{k,L}} = 0. \end{aligned} \tag{5.49}$$

For the ease of notation, we will write H instead of $H_{\bar{s},\bar{t}}(\bar{x},\bar{y})$ until the end of the proof.

In the first step, we work under the polymer measure $\widetilde{P}_{L,\beta}$ and we restrict the trajectories of $\widetilde{\Omega}_L$ to those having an horizontal extension $\approx a(\beta)\sqrt{L}$ and an algebraic area $A_N(V_l) \approx L - a(\beta)\sqrt{L}$. With the second step, we use the random walk representation in Section 3.1 to switch from $\widetilde{P}_{L,\beta}$ to $\mathbf{P}_\beta$. Finally, in steps 3 and 4, we apply the local limit theorem stated in Lemma 5.8 to complete the proof.

Step 1

For $L \in \mathbb{N}$, we rewrite $\psi_{1,L}$ under the form

$$\psi_{1,L}(\bar{x},\bar{y}) = \frac{A_{1,L}}{B_{1,L}} := \frac{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}(H)}{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}}, \tag{5.50}$$

where we recall that $K_L := L + [-\varepsilon(L), \varepsilon(L)] \cap \mathbb{N}$ with $\varepsilon(L) := (\log L)^6$ and where, for $B \subset \widetilde{\Omega}_L$, the quantity $\widetilde{Z}_{L',\beta}(B)$ is the restriction of the partition function $\widetilde{Z}_{L',\beta}$ to those trajectories $l \in B \cap \Omega_{L'}$, i.e.,

$$\widetilde{Z}_{L',\beta}(B) = e^{-\beta L'} \sum_{N=1}^{L'} \sum_{l\in\mathcal{L}_{N,L'}\cap B} e^{\beta \sum_{i=1}^{N-1}(l_i \,\widetilde{\wedge}\, l_{i+1})}.$$

At this stage we set $\eta = \eta_L = L^{-1/8}$ as in Lemma 5.4. We will note $I_{\eta,L} = \big\{(a(\beta) - \eta_L)\sqrt{L}, \dots, (a(\beta)+\eta_L)\sqrt{L}\big\}$ and we introduce the first intermediate function sequence ψ_2, defined as

$$\psi_{2,L}(\bar{x},\bar{y}) = \frac{A_{2,L}}{B_{2,L}} := \frac{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}(H \cap \mathcal{A}_{L,L',\eta})}{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}(\mathcal{A}_{L,L',\eta})}, \tag{5.51}$$

where

$$\mathcal{A}_{L,L',\eta} = \bigcup_{N\in I_{\eta,L}} \big\{l \in \mathcal{L}_{N,L'} : \ |A_N(V_l)| \in L' + [-N - c(\log L)^4, -N] \cap \mathbb{N}\big\}.$$

For simplicity, we will omit the L,η dependency of $\mathcal{A}_{L,L',\eta}$ in what follows. The equivalence $\psi_1 \sim \psi_2$ will be proven once we show that (i) and (ii) in (5.49) are satisfied with $j = 1, k = 2$. We note that

$$\frac{|A_{1,L}-A_{2,L}|}{B_{1,L}} = \frac{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}(H \cap \mathcal{A}^c_{L'})}{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}} \le \frac{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta} P_{L',\beta}(\mathcal{A}^c_{L'})}{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}}, \tag{5.52}$$

with $\mathcal{A}_{L'}^c = \Omega_{L'} \setminus \mathcal{A}_{L'}$. We recall that $K_L = L + [-\varepsilon(L), \varepsilon(L)] \cap \mathbb{N}$ and we use, on the one hand, Lemma 5.4 and the convergence $\lim_{L\to\infty} \frac{\varepsilon(L)}{L} = 0$ to claim that for all $\eta > 0$

$$\lim_{L\to\infty} L^{\frac{r_1+r_2}{2}} \sup_{L'\in K_L} P_{L',\beta}(N_l \notin I_{\eta,L}) = 0, \tag{5.53}$$

and, on the other hand, Lemma 5.5 and $\lim_{L\to\infty} \frac{\varepsilon(L)}{L} = 0$ to assert that there exists $c > 0$ such that

$$\lim_{L\to\infty} L^{\frac{r_1+r_2}{2}} \sup_{L'\in K_L} P_{L',\beta}(|A_{N_l}(V_l)| \notin L' + [-N_l - c(\log L)^4, -N_l]) = 0, \tag{5.54}$$

We combine (5.53) and (5.54) to claim that

$$\lim_{L\to\infty} L^{\frac{r_1+r_2}{2}} \sup_{L'\in K_L} P_{L',\beta}(\mathcal{A}_{L'}^c) = 0. \tag{5.55}$$

We note that (5.52) and (5.55) are sufficient to prove (i). Therefore, it remains to show that (ii) is satisfied. To that aim, we remark that

$$\begin{aligned} \frac{A_{2,L}}{B_{1,L}} \frac{|B_{1,L} - B_{2,L}|}{B_{2,L}} &= \frac{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}(H \cap \mathcal{A}_{L'})}{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}(\mathcal{A}_{L'})} \frac{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}(\mathcal{A}_{L'}^c)}{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}}, & (5.56)\\ &\leq \frac{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta} P_{L',\beta}(\mathcal{A}_{L'}^c)}{\sum_{L'\in K_L} \widetilde{Z}_{L',\beta}}, & (5.57) \end{aligned}$$

and then, we can use directly (5.55) to obtain (ii) and this completes the proof of step 1.

Step 2

To begin with, we set $J_{N,L',L} = \{L' - N - c(\log L)^4, \dots, L' - N\}$ and we note that, with the help of the random walk representation, we can rewrite

$$\begin{aligned} &\psi_{2,L}(\bar{x}, \bar{y}) & (5.58)\\ &= \frac{\sum_{L'\in K_L} \sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in J_{N,L',L}} \mathbf{P}_\beta\big(H, |A_N(V)| = b, V_{N+1} = 0, G_N(V) = L' - N\big)}{\sum_{L'\in K_L} \sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in J_{N,L',L}} \mathbf{P}_\beta\big(|A_N(V)| = b, V_{N+1} = 0, G_N(V) = L' - N\big)}. \end{aligned}$$

We switch the order of summation in (5.58) and we obtain

$$\psi_{2,L}(\bar{x}, \bar{y}) = \frac{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(H, |A_N(V)| = b, V_{N+1} = 0,\, G_N(V) \in U_{b,N,L}\big)}{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(|A_N(V)| = b, V_{N+1} = 0, G_N(V) \in U_{b,N,L}\big)}, \tag{5.59}$$

with

$$\begin{aligned} D_{N,L} &= \{L - \varepsilon(L) - N - c(\log L)^4, \dots, L + \varepsilon(L) - N\}, & (5.60)\\ U_{b,N,L} &= \{b \vee [L - \varepsilon(L) - N], \dots, [b + c(\log L)^4] \wedge [L + \varepsilon(L) - N]\}. & (5.61) \end{aligned}$$

We define the third intermediate function sequence

$$\begin{aligned} \psi_{3,L}(\bar{x}, \bar{y}) &= \frac{A_{3,L}}{B_{3,L}} & (5.62)\\ &:= \frac{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(H, |A_N(V)| = b, V_{N+1} = 0, G_N(V) \in \widetilde{U}_{b,L}\big)}{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(|A_N(V)| = b, V_{N+1} = 0, G_N(V) \in \widetilde{U}_{b,L}\big)}, \end{aligned}$$

with

$$\widetilde{U}_{b,L} = \{b, \dots, b + c(\log L)^4\}. \tag{5.63}$$

The equivalence $\psi_2 \sim \psi_3$ will be proven once we show that (i) and (ii) in (5.49) are satisfied with $j = 3, k = 2$.

For (i), we note that for all $b \in D_{N,L}$ satisfying $b \geq L - \varepsilon(L) - N$ and $b \leq L + \varepsilon(L) - N - c(\log L)^4$ we have $U_{b,L} = \widetilde{U}_{b,L}$ and therefore

$$\begin{aligned}
\frac{|A_{2,L} - A_{3,L}|}{B_{3,L}} &\leq \frac{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(H, |A_N(V)| = b, V_{N+1} = 0, G_N(V) \in \widetilde{U}_{b,L}\big)}{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(|A_N(V)| = b, V_{N+1} = 0, G_N(V) \in \widetilde{U}_{b,L}\big)}, \\
&= \frac{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)\, \mathbf{P}_\beta\big(H \,|\, W_{N,L,b}\big)}{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)},
\end{aligned} \tag{5.64}$$

with

$$\begin{aligned}
\widetilde{D}_{N,L} &= \{b \in D_{N,L} \colon b < L - \varepsilon(L) - N \text{ or } b > L + \varepsilon(L) - N - c(\log L)^4\}, \\
W_{N,L,b} &= \{|A_N(V)| = b,\, V_{N+1} = 0,\, G_N(V) \in \widetilde{U}_{b,L}\}.
\end{aligned} \tag{5.65}$$

For (ii), in turn, with the help of (5.62) we note that,

$$\frac{A_{2,L}}{B_{3,L}} \leq \frac{A_{3,L}}{B_{3,L}} = \frac{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)\, \mathbf{P}_\beta\big(H \,|\, W_{N,L,b}\big)}{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)}, \tag{5.66}$$

and

$$\begin{aligned}
\frac{|B_{2,L} - B_{3,L}|}{B_{2,L}} &\leq \frac{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta(|A_N(V)| = b, V_{N+1} = 0, G_N(V) \in \widetilde{U}_{b,L})}{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in D_{N,L}} \mathbf{P}_\beta(|A_N(V)| = b, V_{N+1} = 0, G_N(V) \in U_{b,L})}, \\
&\leq \frac{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)}{\sum_{N \in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b \in D_{N,L} \setminus \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)},
\end{aligned} \tag{5.67}$$

where we have used again that $\widetilde{U}_{b,L} = U_{b,L}$ for $b \in D_{N,L} \setminus \widetilde{D}_{N,L}$. At this stage, we state two claims that will be sufficient to complete this step.

Claim 5.11. *For $\eta > 0$, there exists a $C > 0$ such that*

$$\limsup_{L \to \infty} \sup_{N \in I_{\eta,L}} \sup_{b \in D_{N,L}} \sup_{(\bar{x}, \bar{y}) \in \mathbb{Z}^{r_1 + r_2}} L^{\frac{r_1 + r_2}{2}} \mathbf{P}_\beta\big(H \,|\, W_{N,L,b}\big) \leq C.$$

Claim 5.12. *For $\eta > 0$, we have*

$$\lim_{L \to \infty} \sup_{N \in I_{\eta,L}} \frac{\sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)}{\sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)} = 0. \tag{5.68}$$

Claims 5.11 and 5.12, together with (5.64), easily imply that (i) is satisfied. Moreover, (5.66) and Claim 5.11 allow us to state that

$$\limsup_{L \to \infty} \sup_{(\bar{x}, \bar{y}) \in \mathbb{Z}^{r_1 + r_2}} L^{\frac{r_1 + r_2}{2}} \tfrac{A_{2,L}}{B_{3,L}} \leq C, \tag{5.69}$$

while Claim 5.12 and (5.67) yield that $\lim_{L \to \infty} \frac{|B_{2,L} - B_{3,L}|}{B_{2,L}} = 0$ which proves (ii) and completes the proof of $\psi_2 \sim \psi_3$.

It remains to display a proof for Claims 5.11 and 5.12.

Proof of Claim 5.11

Since $\eta_L \to 0$, we easily infer that for L large enough and for $N \in I_{\eta,L}$ and $b \in D_{N,L}$ we have

$$\frac{b}{N^2} \in \Big[\frac{1}{2a(\beta)^2}, \frac{2}{a(\beta)^2}\Big] := [R_1, R_2].$$

Thus, we can use Lemma 5.8 and (5.21) to assert that, for L large enough, $L^{\frac{r_1+r_2}{2}} \mathbf{P}_\beta\big(H \,|\, W_{N,L,b}\big)$ is bounded above uniformly in $N \in I_{\eta,L}$ and $b \in \widetilde{D}_{N,L}$. The β dependency of R_1 and R_2 is omitted for simplicity.

Proof of Claim 5.12

By using again the fact that for $N \in I_{\eta,L}$ and $b \in D_{N,L}$ we have, for L large enough, that $\frac{b}{N^2} \in [R_1, R_2]$, we can apply Lemma 5.7, to assert that for L large enough

$$\inf_{N \in I_{\eta,L}} \inf_{b \in D_{N,L}} \mathbf{P}_\beta(G_N(V) \le b + c(\log L)^4 \,\big|\, |A_N(V)| = b, V_{N+1} = 0) \ge \frac{1}{2}. \tag{5.70}$$

We recall the definition of $W_{N,L,b}$ in (5.65) and we set $T_{N,b} := \{|A_N(V)| = b, V_{N+1} = 0\}$. We can bound from above the ratio in the l.h.s. of (5.68) as

$$\frac{\sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)}{\sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)} \le \frac{\sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(T_{N,b}\big)}{\sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)} \le 2\, \frac{\sum_{b \in \widetilde{D}_{N,L}} \mathbf{P}_\beta\big(T_{N,b}\big)}{\sum_{b \in D_{N,L}} \mathbf{P}_\beta\big(T_{N,b}\big)}, \tag{5.71}$$

where we have used (5.70) to obtain that for L large enough, $N \in I_{\eta,L}$ and $b \in D_{N,L}$ we have

$$\mathbf{P}_\beta\big(W_{N,L,b}\big) \ge \mathbf{P}_\beta\big(T_{N,b}\big)\big(\mathbf{P}_\beta\big[G_N(V) \le b + c(\log L)^4 | T_{N,b}\big]\big) \ge \tfrac{1}{2} \mathbf{P}_\beta\big(T_{N,b}\big).$$

At this stage, we need to use the fact that $\varepsilon(L) = \log(L)^6$ and we set

$$d^-_{N,L} = L - \varepsilon(L) - N \quad \text{and} \quad d^+_{N,L} = L + \varepsilon(L) - N,$$

and also

$$D^1_{N,L} = \{d^-_{N,L}, \dots, d^-_{N,L} + (\log L)^5\} \quad \text{and} \quad D^2_{N,L} = \{d^+_{N,L} - (\log L)^5, \dots, d^+_{N,L}\}, \tag{5.72}$$

$$\widetilde{D}^1_{N,L} = \{d^-_{N,L} - c(\log L)^4, \dots, d^-_{N,L}\} \quad \text{and} \quad \widetilde{D}^2_{N,L} = \{d^+_{N,L} - c(\log L)^4, \dots, d^+_{N,L}\}, \tag{5.73}$$

such that $D^1_{N,L}$ and $D^2_{N,L}$ are disjoint subsets of $D_{N,L}$ and $\widetilde{D}^1_{N,L} \cup \widetilde{D}^2_{N,L}$ is a partition of $\widetilde{D}_{N,L}$. We note that all $b \in D^1_{N,L} \cup \widetilde{D}^1_{N,L}$ satisfies $|b - d^-_{N,L}| \le (\log L)^5$ and similarly that all $b \in D^2_{N,L} \cup \widetilde{D}^2_{N,L}$ satisfies $|b - d^+_{N,L}| \le (\log L)^5$. In fact, for L large enough, all the numbers $\frac{b}{N^2}, \frac{d^-_{N,L}}{N^2}, \frac{d^+_{N,L}}{N^2}$ belong to the compact $[R_1, R_2]$ on which the function ρ_β is differentiable. Therefore, there exists a $C > 0$ such that

$$|\rho_\beta(\tfrac{b}{N^2}) - \rho_\beta(\tfrac{d^-_{N,L}}{N^2})| \le C \frac{1}{N^2} (\log L)^5, \quad b \in D^1_{N,L} \cup \widetilde{D}^1_{N,L}, \tag{5.74}$$

$$|\rho_\beta(\tfrac{b}{N^2}) - \rho_\beta(\tfrac{d^-_{N,L}}{N^2})| \le C \frac{1}{N^2} (\log L)^5, \quad b \in D^2_{N,L} \cup \widetilde{D}^2_{N,L}. \tag{5.75}$$

Thus, we can apply Lemma 5.6 to assert that there exists $M_1 > M_2 > 0$ such that for L large enough and for $N \in I_{\eta,L}$ we have

$$\begin{aligned}
&\frac{M_2}{N^2} e^{-N \rho_\beta\left(\frac{d^-_{N,L}}{N^2}\right)} \le \mathbf{P}_\beta(T_{N,b}) \le \frac{M_1}{N^2} e^{-N \rho_\beta\left(\frac{d^-_{N,L}}{N^2}\right)}, \quad b \in D^1_{N,L} \cup \widetilde{D}^1_{N,L} \\
&\frac{M_2}{N^2} e^{-N \rho_\beta\left(\frac{d^+_{N,L}}{N^2}\right)} \le \mathbf{P}_\beta(T_{N,b}) \le \frac{M_1}{N^2} e^{-N \rho_\beta\left(\frac{d^+_{N,L}}{N^2}\right)}, \quad b \in D^2_{N,L} \cup \widetilde{D}^2_{N,L}.
\end{aligned} \tag{5.76}$$

It suffices to combine (5.71) and (5.76) and to note that

$$|\widetilde{D}^1_{N,L}|/|D^1_{N,L}| = |\widetilde{D}^2_{N,L}|/|D^2_{N,L}| = c(\log L)^4/(\log L)^5,$$

to complete the proof of Claim 5.12. Finally, we have proven that $\psi_3 \sim \psi_4$.

Step 3

In this step, we note first that for $L \in \mathbb{N}$, $\psi_{3,L}$ can be written as

$$\psi_{3,L}(\bar{x},\bar{y}) = \frac{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)\, \mathbf{P}_\beta\big(H \,|\, W_{N,L,b}\big)}{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N\, \mathbf{P}_\beta\big(W_{N,L,b}\big)}, \tag{5.77}$$

and we set

$$\psi_{4,L}(\bar{x},\bar{y}) = \frac{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)\, N^{-\frac{r_1+r_2}{2}} \hat{f}_{\widetilde{H}(\frac{b}{N^2},0),\bar{t}}\left(\frac{\bar{y}}{N^{1/2}}, N^{1/2}\gamma_q^*\right) g_{\widetilde{H}(\frac{b}{N^2},0),\bar{s}}\left(\frac{\bar{x}}{N^{1/2}}\right)}{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N\, \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)}. \tag{5.78}$$

We have, thanks to Lemma 5.8, the existence of a sequence $R_N \to 0$ such that

$$\sup_{(\bar{x},\bar{y})\in\mathbb{Z}^{r_1+r_2}} L^{\frac{r_1+r_2}{4}} |\psi_{3,L} - \psi_{4,L}| \le L^{\frac{r_1+r_2}{4}} \sup_{N\in I_{\eta,L}} (R_N N^{-\frac{r_1+r_2}{2}}) \to 0\,.$$

Step 4

Obviously we have for $L \in \mathbb{N}$,

$$\psi_{5,L}(\bar{x},\bar{y}) = \frac{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)\, \psi_{5,L}(\bar{x},\bar{y})}{\sum_{N\in I_{\eta,L}} (\Gamma_\beta)^N\, \sum_{b\in D_{N,L}} \mathbf{P}_\beta\big(W_{N,L,b}\big)}\,.$$

Therefore,

$$\begin{aligned}|\psi_{4,L} - \psi_{5,L}| \le \sup_{N\in I_{\eta,L}, b\in D_{N,L}} \Bigg| & m_L^{-\frac{r_1+r_2}{2}} g_{\beta,\bar{s}}\Big(\tfrac{\bar{x}}{\sqrt{m_L}}\Big)\, f_{\beta,\bar{t}}\Big(\tfrac{\bar{y}}{\sqrt{m_L}}, \sqrt{m_L}\gamma_\beta^*\Big) \\ & - N^{-\frac{r_1+r_2}{2}} \hat{f}_{\widetilde{H}(\frac{b}{N^2},0),\bar{t}}\left(\frac{\bar{y}}{N^{1/2}}, N^{1/2}\gamma_q^*\right) g_{\widetilde{H}(\frac{b}{N^2},0),\bar{s}}\left(\frac{\bar{x}}{N^{1/2}}\right)\Bigg|\,.\end{aligned}$$

Obviously, since $\eta = \eta_L \to 0$,

$$\lim_{L\to\infty} L^{\frac{r_1+r_2}{4}} \sup_{N\in I_{\eta,L}} \left| m_L^{-\frac{r_1+r_2}{2}} - N^{-\frac{r_1+r_2}{2}} \right| = 0\,.$$

By the implicit function theorem applied to the definition (5.10) of $\widetilde{H}$, the function $q \to \widetilde{H}(q,0)$ is globally Lipschitz on compact sets, and thus there exists a constant $C > 0$ such that

$$\sup_{L\ge L_0} \eta_L^{-1} \sup_{N\in I_{\eta,L}, b\in D_{N,L}} \left|\widetilde{H}(q_\beta,0) - \widetilde{H}(\tfrac{b}{N^2},0)\right| \le C\,.$$

By the global Lipschitz properties in $(H,\bar{x},\bar{y})$ of the Gaussian densities $f^c_{H,\bar{t}}(\bar{y})$ and $g_{H,\bar{s}}(\bar{x})$ we conclude that $\psi_4 \sim \psi_5$.

5.6 Proof of Proposition 5.3

Recall (2.14). The proof of the tightness of the sequence of distributions $(\widehat{Q}_{L,\beta})_{L\geq 1}$ is obtained by combining arguments of [14, Section 6] with the steps 1, 2 and 3 of the proof of Proposition 5.2. Let us consider the first coordinate of the process, i.e., $(\sqrt{N_l}\widetilde{M}_l(s))_{s\in[0,1]}$ (the second coordinate is easier to handle since it is even closer to the process studied in [14, Section 6]).

Let us denote by $(\widehat{M}_l(s))_{s\in[0,1]}$ the polygonal interpolation of the middle line. Then, see for example [11] proof of Lemma 5.1.4, the distribution under $\widetilde{P}_{L,\beta}$ of $(\widehat{M}_l(s))_{s\in[0,1]}$ and $(\widetilde{M}_l(s))_{s\in[0,1]}$ are exponentially close, and so we shall restrict ourselves to proving the tightness of the sequence of continuous processes $(\sqrt{N_l}\widehat{M}_l(s))_{s\in[0,1]}$ using the criterion of Theorem 7.3, (ii) of [3]:

$$\begin{aligned}&\forall\varepsilon>0,\eta>0,\exists L_0\in\mathbb{N},\exists\delta\in(0,1)\,,\\ &\widetilde{P}_{L,\beta}\Big(w(\sqrt{N_l}\widehat{M}_l(s),0\leq s\leq 1;\delta)\geq\varepsilon\Big)\leq\eta\qquad(\forall L\geq L_0)\,.\end{aligned}\tag{5.79}$$

where $w(f,\delta)=\sup_{|x-y|\leq\delta}|f(x)-f(y)|$ denotes the modulus of continuity.

We then inspect closely the steps 1, 2 and 3 taken in Proposition 5.2. It is tedious, but straightforward, to see that we only need to prove that

$$\begin{aligned}&\forall\varepsilon>0,\forall\eta>0,\,\exists N_0\in\mathbb{N},\exists\delta\in(0,1)\,,\\ &\mathbf{P}_\beta\Big(w(\tfrac{1}{\sqrt{N}}M_{\lfloor Ns\rfloor},0\leq s\leq 1;\delta)\geq\varepsilon\Big|A_N=qN^2,V_N=0\Big)\leq\eta\qquad(\forall N\geq N_0,\forall q\in[q_1,q_2])\,.\end{aligned}\tag{5.80}$$

To this end, we shall use Kolmogorov's tightness criterion (see Theorem 1.8 Chap. XIII of [30]) and show that

$$\sup_{N\geq N_0,q\in[q_1,q_2],0\leq s,t\leq 1}\mathbf{E}_\beta\left[\left|\frac{M_{\lfloor Ns\rfloor}-M_{\lfloor Nt\rfloor}}{\sqrt{N}}\right|^4\;\Big|\;A_N=qN^2,V_N=0\right]<C\,|t-s|^{\frac{7}{4}}\,.\tag{5.81}$$

At this stage, the proof is completed by mimicking the rather long proof of the weak compactness exposed in [14, Section 6]. For this reason, we briefly sketch the proof below by insisting on those points that need to be slightly improved in [14, Section 6] to achieve (5.81).

The first step, which corresponds to Lemma 6.2 in [14] consists in controlling the supremum in the l.h.s. in (5.81) for $s<t$ satisfying $t-s\leq 1/N^{8/9}$. The key point here consists in controlling, under the conditioning $A_N=qN^2,V_N=0$, the small exponential moments of the increments of V. More precisely we aim at proving that there exists $\eta>0$, $M>0$ and $N_0\in\mathbb{N}$ such that

$$\sup_{q\in[q_1,q_2]}\sup_{N\geq N_0}\sup_{1\leq i\leq N}\mathbf{E}_\beta\left[e^{\eta|U_i|}\;\Big|\;A_N=qN^2,V_N=0\right]\leq M.\tag{5.82}$$

In [14] such result is displayed in Lemma 5.3 but without the supremum in $q\in[q_1,q_2]$. As in the proof of Theorem 5.10, we overstep this difficultly with the help of Proposition 5.1 which ensures us that the tilting parameters H_N^q remain in a compact subset of $\mathcal{D}$ when $q\in[q_1,q_2]$ and N is large enough.

In the second step, we must deal with $s<t$ satisfying $t-s\geq 1/N^{8/9}$ which corresponds to Lemma 6.3 in [14]. We set $\xi_N^{s,t}:=M_{\lfloor Nt\rfloor}-M_{\lfloor Ns\rfloor}$ and

$$e_N^{s,t}:=\frac{1}{N}E_{N,H_N^q}(\xi_N^{s,t})=\frac{1}{N}\sum_{i=\lfloor Ns\rfloor}^{\lfloor Nt\rfloor}(-1)^{i-1}\mathfrak{L}'(h_{N,i}),\tag{5.83}$$

with $h_{N,i} = (1-\frac{i}{N})h_0^{N,q} + h_1^{N,q}$. The strategy consists in proving the uniform boundedness of both

$$\mathbf{E}_\beta\left[\left|\frac{\xi_N^{s,t}-Ne_N^{s,t}}{\sqrt{N(t-s)}}\right|^4 \;\Big|\; A_N = qN^2, V_N = 0\right] \quad \text{and} \quad \frac{|\sqrt{N}e_N^{s,t}|}{\sqrt{t-s}}, \tag{5.84}$$

which is enough to complete the proof of (5.81).

The second term in (5.84) is dealt with by using again inequality (5.46). We conclude that there exists a $C > 0$ such that $|e_N^{s,t}| \leq C(t-s)/N$ for N large enough, for $s < t$ and for $q \in [q_1, q_2]$ which is more than what we need.

At this stage, it remains to control the first term in (5.84), which can be rewritten as

$$\int_0^\infty u^3\, \mathbf{P}_\beta\left(\left|\tfrac{\xi_N^{s,t}-Ne_N^{s,t}}{\sqrt{N(t-s)}}\right| > u \;\Big|\; A_N = qN^2, V_N = 0\right) du. \tag{5.85}$$

The latter is achieved by refining the tilting procedure introduced in (5.3–5.10) in such a way that the event $\left\{\left|\frac{\xi_N^{s,t}-Ne_N^{s,t}}{\sqrt{N(t-s)}}\right| > u\right\}$ is integrated in the tilting. This tilting, combined with Propostion 5.1, allows us to bound from above the probability under the integral in (5.85) by Ce^{-cu^2} uniformly in N, $s<t$, $q \in [q_1, q_2]$ and $u \in [0, N^{1/19}]$ and by $CN^2e^{-cu^2}$ for $u \in [N^{1/19}, \infty)$ which suffices to complete the proof.

5.7 Proof of Lemmas 5.4–5.7

Proof of Lemma 5.4

Let us recall the notation $I_{j_{\max}}$ of [10, Section 1] which is the set of indexes of stretches that occur in the largest bead. In other word it is the largest set of consecutive indices for which $V_{l,i}$ keeps the same sign. To begin with, we can show, following *mutatis mutandis* the proof of Theorem C, that for $\alpha > 0$, there exists a $c > 0$ such that

$$\lim_{L\to\infty} L^\alpha\, P_{L,\beta}\big[I_{j_{\max}} \leq L - c(\log L)^4\big] = 0. \tag{5.86}$$

Then, recall form section 4.4 of [10], that $a(\beta)$ is the unique maximum of the strictly concave function

$$\widetilde{G}(a) := a \log \Gamma_\beta - \tfrac{1}{a}\, \widetilde{h}_0\big(\tfrac{1}{a^2}, 0\big) + a\mathfrak{L}_\Lambda\big(\widetilde{H}\big(\tfrac{1}{a^2}, 0\big)\big), \tag{5.87}$$

Therefore, inequality (4.50) of [10] taken with $\varepsilon' = \eta_L = L^{-\frac{1}{8}}$ yields that for any $\alpha > 0$

$$\lim_{L\to\infty} L^\alpha\, P_{L,\beta}\left(\left(B_{\eta_L,L}\right)^c \cap \left\{I_{j_{\max}} \leq L - c(\log L)^4\right\}\right) = 0\,.$$

Hence we get the desired result by combining the last result and (5.86).

Remark 5.13. We can go a step further in our inspection of the latter proof and establish that the distribution of $\frac{N_L}{\sqrt{L}}$ under $P^o_{L,\beta}$, the polymer measure restricted to have only one bead, follows a large deviation principle with good rate function $a \to \widetilde{G}(a(\beta)) - \widetilde{G}(a)$. However we are unable to prove the same LDP under $P_{L,\beta}$.

Proof of Lemma 5.5

Note that for $l \in \Omega_L$, we have $\sum_{i=1}^{N_l} |V_{l,i}| = \sum_{i=1}^{N_l} |l_i| = L - N_l$ and thus, by the definition of $I_{j_{\max}}$ we have also

$$\sum_{i=1}^{N_l} |V_{l,i}| - 2\sum_{i\notin I_{\max}} |V_{l,i}| \leq \Big|\sum_{i=1}^{N_l} V_{l,i}\Big| \leq \sum_{i=1}^{N_l} |V_{l,i}|. \tag{5.88}$$

Moreover, we note that $l \in \{I_{j_{\max}} \geq L - c(\log L)^4\}$ yields

$$\sum_{i \notin I_{j_{\max}}} |V_{l,i}| = \sum_{i \notin I_{j_{\max}}} |l_i| \leq c(\log L)^4. \tag{5.89}$$

At this stage, we recall that $A_N(V) = \sum_{i=1}^N V_i$ and we use (5.88) and (5.89) to assert that $l \in \{I_{j_{\max}} \geq L - c(\log L)^4\}$ implies $\big|A_{N_l}(V_l)\big| \in [L - N_l - 2c(\log L)^4, L - N_l]$. It remains to use (5.86) to complete the proof of Lemma 5.5.

Proof of Lemma 5.7

For the sake of conciseness, we will use, in this proof only, the notations

$$\begin{aligned} W_{c,N,q} &= \{G_N(V) \geq qN^2 + c(\log N)^4,\, V_N = 0,\, A_N(V) = qN^2\}, \\ T_{N,q} &= \{V_N = 0,\, A_N(V) = qN^2\}, \end{aligned} \tag{5.90}$$

with $c > 0$ and $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$. Thus, we can restate (5.19) under the form

$$\lim_{N \to \infty} \sup_{q \in [q_1, q_2]} N^\alpha \mathbf{P}_\beta\big(W_{c,N,q} \,|\, T_{N,q}\big) = 0, \tag{5.91}$$

where the intersection of $[q_1, q_2]$ with $\frac{\mathbb{N}}{N^2}$ is omitted for simplicity. Since the equality $P_{N,H_N^q}\big(W_{c,N,q} \,|\, T_{N,q}\big) = \mathbf{P}_\beta\big(W_{c,N,q} \,|\, T_{N,q}\big)$ holds for all $N \in \mathbb{N}$ and $q \in [q_1, q_2]$, we can use [10, Proposition 2.2] to ensure that (5.91) will be proven once we show that for $c, \alpha > 0$ we have

$$\lim_{N \to \infty} \sup_{q \in [q_1, q_2]} N^\alpha P_{N,H_N^q}\big(W_{c,N,q}\big) = 0. \tag{5.92}$$

For $N \in \mathbb{N}$ and $\widetilde{c} > 0$, we let $U_{N,\widetilde{c}}$ be the set containing those trajectories that do not remain strictly positive on the interval $[\widetilde{c} \log N, N - \widetilde{c} \log N] \cap \mathbb{N}$ and satisfy $V_N = 0$, i.e.,

$$\begin{aligned} U_{N,\widetilde{c}} = \{\max\{i \leq \tfrac{N}{2} :\, V_i \leq 0\} \geq \widetilde{c} \log N,\, V_N = 0\} \\ \cup \{\max\{i \leq \tfrac{N}{2} :\, V_{N-i} \leq 0\} \geq \widetilde{c} \log N,\, V_N = 0\}, \end{aligned} \tag{5.93}$$

so that we can write the upper bound

$$P_{N,H_N^q}\big(W_{c,N,q}\big) \leq P_{N,H_N^q}(U_{N,\widetilde{c}}) + P_{N,H_N^q}(W_{c,N,q} \cap (U_{N,\widetilde{c}})^c). \tag{5.94}$$

At this stage, we need to distinguish between the positive part $A_N^+(V)$ and the negative part $A_N^-(V)$ of the algebraic area below the V trajectory, i.e.,

$$A_N^-(V) = -\textstyle\sum_{i=1}^N V_i \, 1_{\{V_i \leq 0\}} \quad \text{and} \quad A_N^+(V) = \textstyle\sum_{i=1}^N V_i, 1_{\{V_i \geq 0\}}.$$

As a consequence, the geometric and the algebraic areas below the V trajectory can be written as $A_N(V) = A_N^+(V) - A_N^-(V)$ and $G_N(V) = A_N^+(V) + A_N^-(V)$ and therefore, under the event $A_N(V) = qN^2$ we have $G_N(V) = qN^2 + 2A_N^-(V)$. Thus, under the event $W_{c,N,q} \cap (U_{N,\widetilde{c}})^c$ we have necessarily that $A_N^-(V) \geq \frac{c}{2}(\log N)^4$ and since V is strictly positive between $\widetilde{c} \log N$ and $N - \widetilde{c} \log N$ we can write

$$\begin{aligned} W_{c,N,q} \cap (U_{N,\widetilde{c}})^c \,\subset\, \big\{A^-_{\widetilde{c} \log N}(V) \geq \tfrac{c}{4}(\log N)^4,\, V_N = 0\big\} \\ \cup \big\{A^-_{N - \widetilde{c} \log N, N}(V) \geq \tfrac{c}{4}(\log N)^4,\, V_N = 0\big\}\,, \end{aligned} \tag{5.95}$$

where $A^-_{N-\widetilde{c}\log N, N} = -\sum_{i=N-\widetilde{c}\log N}^N V_i 1_{\{V_i \geq 0\}}$. Moreover, under P_{N,H_N^q}, the sequences of random variables $(U_i)_{i=1}^N$ and $(-U_{N-i})_{i=0}^{N-1}$ have the same law (by symmetry) and therefore the equality

$$P_{N,H_N^q}\big((V_1, \dots, V_k) \in A, V_N = 0\big) = P_{N,H_N^q}\big((V_{N-1}, \dots, V_{N-k}) \in A, V_N = 0\big) \tag{5.96}$$

holds true for all $k \in \{1, \dots, N-1\}$ and $A \in \mathrm{Bor}(\mathbb{R}^k)$. A straightforward application of (5.96) tells us that under P_{N,H_N^q}, both sets in the r.h.s. of (5.93) have the same probability and similarly both sets in the r.h.s. of (5.95) have the same probability. Therefore we can combine (5.93), (5.94) and (5.95) to obtain

$$P_{N,H_N^q}\big(W_{N,q}\big) \leq 2P_{N,H_N^q}\big(\exists i \in \{\widetilde{c}\log N, \dots, \tfrac{N}{2}\}\colon V_i \leq 0\big) \\ + 2P_{N,H_N^q}\big(A^-_{\widetilde{c}\log N}(V) \geq \tfrac{c}{4}(\log N)^4\big). \tag{5.97}$$

As a consequence, the proof of Lemma 5.7 will be complete once we show, on the one hand, that for all $\alpha > 0$ there exists a $\widetilde{c} > 0$ such that

$$\lim_{N\to\infty} \sup_{q\in[q_1,q_2]} N^\alpha P_{N,H_N^q}\big(\exists i \in \{\widetilde{c}\log N, \dots, \tfrac{N}{2}\}\colon\ V_i \leq 0\big) = 0, \tag{5.98}$$

and, on the other hand, that for all $\alpha, x, y > 0$ we have

$$\lim_{N\to\infty} \sup_{q\in[q_1,q_2]} N^\alpha P_{N,H_N^q}\big(A^-_{x\log N}(V) \geq y(\log N)^4\big) = 0. \tag{5.99}$$

In order to prove (5.98) and (5.99) we recall Lemma 6.2 in [10],

Lemma 5.14. *For $[q_1, q_2] \subset (0, \infty)$ there exists $N_0 \in \mathbb{N}$ and there exist three positive constants C', C_1, λ such that for $N \geq N_0$ and for every integer $j \leq N/2$, the following bound holds*

$$\mathbf{E}_{N,H_N^q}\big[e^{-\lambda V_j}\big] \leq C' e^{-C_1 j}, \quad N \in \mathbb{N}. \tag{5.100}$$

To prove (5.98), we apply Lemma 5.14 directly and we obtain

$$P_{N,H_N^q}(\exists i \in \{\widetilde{c}\log N, \dots, \tfrac{N}{2}\}\colon\ V_i \leq 0) \leq \sum_{j=\widetilde{c}\log N}^{N/2} P_{N,H_N^q}\big(e^{-\lambda V_j} \geq 1\big) \leq C' \sum_{j=\widetilde{c}\log N}^{N/2} e^{-C_1 j}, \tag{5.101}$$

which suffices to complete the proof of (5.98). For the proof of (5.99), we note that

$$\big\{A^-_{x\log N}(V) \geq y(\log N)^4\big\} \subset \big\{\exists i \leq x\log N\colon\ V_i \leq -\tfrac{y}{x}(\log N)^3\big\}, \tag{5.102}$$

and we apply Lemma 5.14 again to obtain

$$P_{N,H_N^q}(\exists i \leq x\log N\colon\ V_i \leq -\tfrac{y}{x}(\log N)^3) \leq \sum_{j=1}^{x\log N} P_{N,H_N^q}\big(e^{-\lambda V_j} \geq e^{\frac{y\lambda}{x}(\log N)^3}\big) \tag{5.103}$$

$$\leq \sum_{j=1}^{x\log N} E_{N,H_N^q}\big(e^{-\lambda V_j}\big)\, e^{-\frac{y\lambda}{x}(\log N)^3} \tag{5.104}$$

$$\leq C' x\log N e^{-\frac{y\lambda}{x}(\log N)^3}, \tag{5.105}$$

and this completes the proof of (5.99).

6 Scaling Limits in the extended phase

As mentioned in the introduction, we will not display the details of the proofs of Theorems 2.1 (1) and 2.2 (1) and of Theorem 2.8. The technique that we use consists in identifying an underlying renewal process based on an ad-hoc decomposition of the path into patterns that are not interacting with each other energetically. Once this associated renewal process is obtained and once we prove that the length of a pattern is integrable, the rest of the proof becomes standard.

We shall restrict ourselves to paths of length L whose last stretch has a zero vertical length, i.e., $\Omega_L^c = \{l \in \Omega_L : l_{N_l} = 0\}$. Note that the natural one-to-one correspondence between Ω_L and Ω_{L+1}^c conserves the Hamiltonian and therefore, proving Theorem 2.1 (1) or 2.2 (1) or Theorem 2.8 with or without the constraint is equivalent.

Let us define a *pattern* as a path whose first zero length vertical stretch occurs only at the end of the path. We shall decompose a path into a finite number of patterns, that is for $l \in \Omega_L^c$ we consider the successive indices corresponding to vertical stretches of zero length, i.e.,

$$T_0 = 0, T_{k+1}(l) = \inf\{j \geq 1 + T_k : l_j = 0\}.$$

Then $\mathfrak{N}_k = T_k - T_{k-1}$ is the horizontal extension of the k-th pattern, $S_k = \mathfrak{N}_k + |l_{T_{k-1}+1}| + \cdots + |l_{T_k}|$ is the length of the k-th pattern and $\mathfrak{J}_k = l_{T_{k-1}+1} + \cdots + l_{T_k}$ is the vertical displacement on the k-th pattern. If $\pi_L(l) = r$ is the number of patterns, then the horizontal extension is $N_l = \mathfrak{N}_1 + \cdots + \mathfrak{N}_r$, the total length is of course $L = S_1 + \cdots + S_r$ and the total vertical displacement is $\mathfrak{J}_1 + \cdots + \mathfrak{J}_r$. The key observation that will lead to the construction of the renewal structure, is that the Hamiltonian of the path is the sum of the Hamiltonian of the patterns, since the separating two horizontal steps prevent any interaction between the patterns.

Let us define the pattern excess partition function $\hat{Z}_{L,\beta}$ and apply the probabilistic representation displayed in (3.2–3.3) to obtain

$$\begin{aligned}\hat{Z}_{L,\beta} := \widetilde{Z}_{L,\beta}(T_1(l) = N_l) &= e^{-\beta L} \sum_{l \in \Omega_L} \mathbf{P}_L(l)\, e^{H_{L,\beta}(l)}\, 1_{\{T_1(l)=N_l\}} \\ &= \sum_{N=1}^{L} (\Gamma_\beta)^N\, \mathbf{P}_\beta\left(G_N(V) = L - N, T(V) = N\right),\end{aligned} \tag{6.1}$$

where, for $V \in \mathbb{Z}^{\mathbb{N}_0}$ such that $V_0 = 0$, we set $T(V) = \inf\{i \geq 1 \colon V_i = 0\}$. For the associated random walk trajectory V, the vertical displacement is given by $Y_n(V) := \sum_{i=1}^{n} (-1)^{i-1} V_i$ for $n \in \mathbb{N}$.

We will use the decomposition into patterns to generate an auxiliary renewal process, whose inter-arrivals are associated with the successive lengths of the patterns. Thus, it is natural to consider the series

$$\varphi(\alpha) := \sum_{t \geq 1} \hat{Z}_{t,\beta} e^{-\alpha t} \in]0, +\infty], \tag{6.2}$$

and the convergence abscissa $\hat{f}(\beta) := \inf\{\alpha : \varphi(\alpha) < +\infty\}$. An important observation at this stage is the link between φ and $\widetilde{f}(\beta)$ that is stated in the following lemma.

Lemma 6.1. *In the extended phase we have* $0 < \hat{f}(\beta) < \widetilde{f}(\beta)$ *and moreover* $\varphi(\widetilde{f}(\beta)) = 1$.

Lemma 6.1 allows us to define rigorously the renewal process. We even enlarge the probability space on which this renewal process is defined to take into account the horizontal extension and the vertical displacement on each pattern. We finally obtain an auxiliary regenerative process that will be the cornerstone of our study of the extended phase. To that aim, we let $(\sigma_i, \nu_i, y_i)_{i \geq 1}$ be an IID sequence of random variables of law $\mathfrak{P}_\beta$. The law of (σ_1, n_1, y_1) is given by

- $\mathfrak{P}_\beta(\sigma_1 = s) = \hat{Z}_{s,\beta}\, e^{-s\widetilde{f}(\beta)}, \quad \text{for } s \geq 1.$

- The conditional distribution of ν_1 given $\sigma_1 = s$ is (recall (6.1))

$$\mathfrak{P}_\beta(\nu_1 = n \mid \sigma_1 = s) = \frac{1}{\hat{Z}_{s,\beta}} (\Gamma_\beta)^n\, \mathbf{P}_\beta\left(G_n(V) = s - n, T(V) = n\right) \qquad (1 \leq n \leq s).$$

- The conditional distribution of y_1 given $\sigma_1 = s, \nu_1 = n$ is

$$\mathfrak{P}_\beta(y_1 = t \mid \sigma_1 = s, \nu_1 = n) = \mathbf{P}_\beta\left(Y_n(V) = t \mid G_n(V) = s-n, T(V) = n\right) \qquad (t \in \mathbb{Z}).$$

The link between the latter regenerative process and the polymer law is stated in Lemma 6.2 below. We let $\mathcal{T}$ be the set of renewal times associated to σ, i.e., $\mathcal{T} = \{\sigma_1 + \dots + \sigma_r, r \in \mathbb{N}\}$.

Lemma 6.2. *Given integers $r, s_1, \dots, s_r, n_1, \dots, n_r, t_1, \dots, t_r$ such that $s_i \geq 1$, $1 \leq n_i \leq s_i$, $s_1 + \dots + s_r = L$, we have*

$$P^c_{L,\beta}\left((S_i, \mathfrak{N}_i, \mathfrak{J}_i) = (s_i, n_i, t_i), 1 \leq i \leq r\right) = \mathfrak{P}_\beta\big((\sigma_i, \nu_i, y_i) = (s_i, n_i, t_i), 1 \leq i \leq r \mid L \in \mathcal{T}\big).$$

As a consequence of Lemma 6.1, the length of a pattern σ_1 is integrable under $\mathfrak{P}_\beta$. Thus, Lemma 6.2 allows us to state that the proofs of Theorems 2.1 (1) or 2.2 (1) and of Theorem 2.8 become standard applications of regenerative process theory, see e.g. Section 5.10 of [31].

7 Appendix

7.1 Perfect simulation procedure

We shall use the acceptance-reject algorithm that we recall briefly. Let $X, X_1, X_2, \dots$ be i.i.d. random variables with values in a measurable set $(E, \mathcal{E})$. Let A be a measurable subset of E such that $\mathbb{P}(X \in A) > 0$ and set

$$T := \inf\{n \geq 1 : X_n \in A\}.$$

Then, the acceptance-reject algorithm ensures that T has a geometric distribution of parameter $\mathbb{P}(X \in A)$ and that X_T is independent of T, with distribution the conditional law

$$\mathbb{P}(X_T \in B) = \mathbb{P}(X \in B \mid X \in A).$$

For $\beta = \beta_c$, we have $\Gamma_\beta = 1$ so the representation formula (1.7) yields

$$Z_{L,\beta} = c_\beta e^{\beta L} \sum_{N=1}^{L} \mathbf{P}_\beta\left(V \in \mathcal{V}_{N,L-N}\right) = c_\beta e^{\beta L} \mathbf{P}_\beta\left(V \in A\right),$$

with $A = \{V : \exists N : G_N(V) = L - N, V_{N+1} = 0\}$.

For $N \in \mathbb{N}$, we recall the definition of T_N in (3.1). We also associate with every trajectory V the index $N_L(V)$ defined as $\inf\{N \geq 0\colon\ G_N(V) \geq L - N\}$. Then, we follow the same steps as in Section 3.1 to show that if B is a set of trajectories in Ω_L we have

$$P_{L,\beta}\left(l \in B\right) = \frac{c_\beta e^{\beta L}}{Z_{L,\beta}} \sum_{N=1}^{L} \mathbf{P}_\beta\left(V \in \mathcal{V}_{N,L-N}, T_N(V) \in B\right) = \mathbf{P}_\beta\left(T_{N_L(V)}(V) \in B \mid V \in A\right). \tag{7.1}$$

Therefore we can use an acceptance-reject to simulate a trajectory V of law $\mathbf{P}_\beta(\cdot \mid V \in A)$ and then apply the transformation $T_{N_L(V)}$ to V to obtain an IPDSAW under $P_{L,\beta}$ (for $\beta = \beta_c$). The mean number of rejects for an acceptance is the mean of the geometric r.v. that is, thanks to Theorem 2.1,

$$\frac{1}{P_{L,\beta}\left(V \in A\right)} = \frac{1}{\widetilde{Z}_{L,\beta}} \sim \frac{1}{c} L^{2/3}.$$

In a nutshell, we have a perfect simulation algorithm with complexity $L^{2/3}$.

Of course, one can try to use the same trick for $\beta \neq \beta_c$, say $\beta > \beta_c$ so that $\Gamma_\beta < 1$ We let S be an independent geometric r.v. of parameter $1 - \Gamma_\beta$, and add a cemetery point δ for trajectories: S is now a lifetime so that $V = \delta$ if $N_L(V) \geq S$. Under this new probability $\bar{\mathbf{P}}_{L,\beta}$ we have

$$P_{L,\beta}(l \in B) = \bar{\mathbf{P}}_{L,\beta}(T_{N_L(V)}(V) \in B \mid V \in A)\,,$$

and we have a perfect simulation algorithm. The problem is that now the mean number of reject for an acceptance is growing very fast with L:

$$\frac{1}{\widetilde{Z}_{L,\beta}} \geq C\frac{1}{L^\kappa} e^{c\sqrt{L}}.$$

7.2 Proof of (4.42)

Since by definition

$$\begin{aligned} &\tau = \inf\left\{i \geq 1 \colon\; V_{i-1} \neq 0 \text{ and } V_{i-1}V_i \leq 0\right\}, \\ &\widetilde{\tau} = \inf\left\{i \geq 1 \colon\; V_i \leq 0\right\}, \end{aligned} \tag{7.2}$$

we can claim that $V_0 > 0$ implies $\tau = \widetilde{\tau}$. Thus, we recall (4.4) and can write

$$\mathbf{P}_{\beta,\mu_\beta}\left(\tau = n\right) = \mu_\beta(0)\,\mathbf{P}_\beta\left(\tau = n\right) + 2\sum_{x=1}^{\infty} \mu_\beta(x)\,\mathbf{P}_{\beta,x}(\widetilde{\tau} = n). \tag{7.3}$$

By disintegrating the event $\{\widetilde{\tau} = n+1\}$ with respect to the value taken by V_1 and by recalling that for $x > 1$ we have $\mathbf{P}_\beta(V_1 = x) = 2\mu_\beta(x)/c_\beta(1 - e^{-\beta/2})$ we obtain that

$$\mathbf{P}_\beta(\widetilde{\tau} = n+1) = \frac{2}{1 + e^{-\beta/2}} \sum_{x=1}^{\infty} \mu_\beta(x)\,\mathbf{P}_{\beta,x}(\widetilde{\tau} = n). \tag{7.4}$$

Moreover, under $\mathbf{P}_\beta$ we can disintegrate the event $\{\tau = n\}$ with respect to the time k during which V sticks to 0 before leaving it, i.e.,

$$\begin{aligned} \mathbf{P}_\beta\left(\tau = n\right) &= 2\sum_{k=0}^{n-2} \mathbf{P}_\beta(V_1 = \cdots = V_k = 0,\, V_{k+1} > 0, \ldots, V_{n-1} > 0, V_n \leq 0), \\ &= 2\sum_{k=0}^{n-2} \frac{1}{(c_\beta)^k}\,\mathbf{P}_\beta(\widetilde{\tau} = n-k). \end{aligned} \tag{7.5}$$

At this stage, we recall that by Theorem 8 of [26] we have that $\mathbf{P}_\beta\left(\widetilde{\tau} = n\right) \sim Cn^{-3/2}$ with $C = (\mathbf{E}_\beta\left[V_1^2\right]/2\pi)^{1/2}$. Then, it remains to recall that $\mu_\beta(0) = 1 - e^{-\beta/2}$ and to put (7.3–7.5) together to obtain

$$\mathbf{P}_{\beta,\mu_\beta}\left(\tau = n\right) \sim \frac{C}{n^{3/2}}\left[2(1 - e^{-\beta/2})\frac{c_\beta}{c_\beta - 1} + 1 + e^{-\beta/2}\right], \tag{7.6}$$

which after a straightforward computation gives us

$$\mathbf{P}_{\beta,\mu_\beta}\left(\tau = n\right) \sim (1 + e^{\beta/2})\frac{C}{n^{3/2}}. \tag{7.7}$$

7.3 Proof of Proposition 4.7

Let $(X_n)_{n\geq 1}$ be an IID sequence of discrete random variables, centered with variance σ^2. Let $S_n = X_1 + \cdots + X_n$ and $A_n = S_1 + \cdots + S_n$. Then, as a consequence of Donsker Theorem, we have the convergence in distribution

$$\left(\frac{S_n}{\sigma\sqrt{n}}, \frac{A_n}{\sigma n^{3/2}}\right) \xrightarrow[n\to+\infty]{d} (B_1, I_1), \tag{7.8}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion and $I_t = \int_0^t B_s\, ds$.

The vector (B_1, I_1) is centered Gaussian with characteristic function

$$\mathbb{E}\left[e^{i(\vartheta_1 B_1 + \vartheta_2 I_1)}\right] = e^{-\frac{1}{2}(\vartheta_1^2 + \vartheta_1\vartheta_2 + \frac{\vartheta_2^2}{3})} = e^{-\frac{1}{2}\Gamma\vartheta.\vartheta},$$

and density $g(u,v)$.

[6] established the following local limit theorem: if

$$p_n(k,a) := \mathbb{P}\left(S_n = k, A_n = a\right), \quad \bar{p}_n(k,a) := \frac{1}{n^2\sigma^2} g\left(\frac{k}{\sigma\sqrt{n}}, \frac{a}{\sigma n^{3/2}}\right) \quad (k, a \in \mathbb{Z}). \tag{7.9}$$

then

$$\sup_{k,a\in\mathbb{Z}} n^2 \left|p_n(k,a) - \bar{p}_n(k,a)\right| \to 0. \tag{7.10}$$

Our goal is to improve this convergence, when X has a fourth moment. More precisely we assume that $\mathbb{E}\left[X_1^4\right] < \infty$ and $\mathbb{E}\left[X_1^3\right] = 0$. Then,

$$\sup_{k,a\in\mathbb{Z}} n^3 \left|p_n(k,a) - \bar{p}_n(k,a)\right| < +\infty\,. \tag{7.11}$$

The techniques used are found in the book [27], and we have decided to keep as much as possible their notations (in particular we shall assume without loss in generality that $\sigma = 1$). By Fourier inversion formula if: $x := (k,a)$, $z := z(n,k,a) = (\frac{k}{\sigma\sqrt{n}}, \frac{a}{\sigma n^{3/2}})$ then

$$p_n(k,a) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} e^{-ix.\vartheta}\psi_n(\vartheta_1,\vartheta_2)\, d\vartheta = \frac{1}{(2\pi)^2 n^2}\int_{A_n} e^{-iz.\vartheta}\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right) d\vartheta, \tag{7.12}$$

with $A_n = \sqrt{n}[-\pi,\pi] \times n^{3/2}[-\pi,\pi]$ and

$$\psi_n(u,v) := \mathbb{E}\left[e^{i(uS_n + vA_n)}\right] = \mathbb{E}\left[\prod_{m=1}^n e^{i(u+mv)X_{n+1-m}}\right] = \prod_{m=1}^n \varphi(u+mv)\,,$$

with $\varphi(u) = \mathbb{E}\left[e^{iuX_1}\right]$. We can choose $\delta > 0$ such that for every $|u| \leq \delta$ we have $|\varphi(u) - 1| \leq \frac{1}{2}$. Then, for $|u| \leq \delta$ we get

$$\log\varphi(u) = -\frac{1}{2}u^2 + h(u),$$

with $h(u) = O(u^4)$ since $\mathbb{E}\left[X_1^3\right] = 0$ and $\mathbb{E}\left[X_1^4\right] <= \infty$. Therefore, if $\vartheta \in \delta A_n$,

$$\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right) = \exp\Big(-\frac{1}{2n}\sum_{m=1}^n (\vartheta_1 + \frac{m}{n}\vartheta_2)^2 + f(\vartheta,n)\Big) = \exp\Big(-\frac{1}{2}\Gamma\vartheta.\vartheta + g(\vartheta,n)\Big)\,, \tag{7.13}$$

with $f(\vartheta,n) = \sum_{m=1}^n h(\frac{1}{\sqrt{n}}((\vartheta_1 + \frac{m}{n}\vartheta_2)$ and thus $|f(\vartheta,n)| \leq C\frac{|\vartheta|^2}{n}$. We need now to evaluate the error we make by approximating the integral by a Riemann sum: if $l_\vartheta(x) = (\vartheta_1 + x\vartheta_2)^2$, then $\Gamma\vartheta.\vartheta = \int_0^1 l_\vartheta(x)dx$ and

$$|g(\vartheta,n) - f(\vartheta,n)| = \frac{1}{2}\left|\frac{1}{n}\sum_{m=1}^n l_\vartheta(\frac{m}{n}) - \int_0^1 l_\vartheta(x)dx\right| \leq C\frac{|\vartheta|^2}{n}\,. \tag{7.14}$$

Since the quadratic form $\Gamma\vartheta.\vartheta$ is positive definite, there exists n_0 such that for $n \geq n_0$ and $\vartheta \in \delta A_n$:

$$|g(\vartheta,n)| \leq \max\Big(\frac{1}{4}\Gamma\vartheta.\vartheta, C\frac{|\vartheta|^2}{n}\Big)\,. \tag{7.15}$$

We know, see [27, Lemma 2.3.3] that there exists $b > 0$ such that

$$|\varphi(u)| \le 1 - bu^2 \le e^{-bu^2} \qquad (u \in [-\pi, \pi])\,. \tag{7.16}$$

Since Γ is positive definite there exists $c > 0$ such that

$$c\,|\vartheta|^2 \le \Gamma\vartheta.\vartheta \le \frac{1}{c}\,|\vartheta|^2\,, \tag{7.17}$$

and thus for every $\vartheta \in A_n$, using (7.14), and $n \ge n_0$,

$$\left|\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right)\right| \le e^{-b\sum_{m=1}^{n}(\frac{\vartheta_1}{\sqrt{n}}+\frac{m}{n^{3/2}}\vartheta_2)^2} \le e^{-b\Gamma\vartheta.\vartheta+O(\frac{|\vartheta|^2}{n})} \le e^{-\frac{1}{2}b\Gamma\vartheta.\vartheta} \le e^{-\frac{bc}{2}|\vartheta|^2}\,. \tag{7.18}$$

We shall split the integral defining $p_n(k,a)$ in two, with a parameter $\eta > 0$ that we shall choose later:

$$\begin{aligned} p_n(k,a) = \frac{1}{(2\pi)^2 n^2}\Bigg(&\int_{A_n\cap\{|\vartheta|\le\eta\sqrt{n}\}} e^{-iz.\vartheta}\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right)d\vartheta \\ &+ \int_{A_n\cap\{|\vartheta|>\eta\sqrt{n}\}} e^{-iz.\vartheta}\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right)d\vartheta\Bigg)\,. \end{aligned} \tag{7.19}$$

The second integral is easily bounded:

$$\begin{aligned} \left|\int_{A_n\cap\{|\vartheta|>\eta\sqrt{n}\}} e^{-iz.\vartheta}\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right)d\vartheta\right| &\le \int_{\{|\vartheta|>\eta\sqrt{n}\}} \left|\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right)\right| d\vartheta \\ &\le \int_{\{|\vartheta|>\eta\sqrt{n}\}} e^{-\frac{bc}{2}|\vartheta|^2} d\vartheta \le Ce^{-\beta n}. \end{aligned} \tag{7.20}$$

Again by Fourier inversion formula,

$$\bar{p}_n(k,a) = \frac{1}{(2\pi)^2 n^2}\int e^{-iz.\vartheta}e^{-\frac{1}{2}\Gamma\vartheta.\vartheta}\,d\vartheta. \tag{7.21}$$

We also split into two this integral and bound the second term:

$$\left|\int_{\{|\vartheta|>\eta\sqrt{n}\}} e^{-iz.\vartheta}e^{-\frac{1}{2}\Gamma\vartheta.\vartheta}\,d\vartheta\right| \le \int_{\{|\vartheta|>\eta\sqrt{n}\}} e^{-\frac{1}{2}c|\vartheta|^2} \le e^{-\beta' n}\,. \tag{7.22}$$

Now choose $\eta > 0$ small enough so that $\{|\vartheta| \le \eta\sqrt{n}\} \subset \delta A_n$. We get, maybe for a different $\beta > 0$,

$$\begin{aligned} p_n(k,a) - \bar{p}_n(k,a) &= O(e^{-\beta n}) + \frac{1}{(2\pi)^2n^2}\int_{\{|\vartheta|\le\eta\sqrt{n}\}} e^{-iz.\vartheta}(\psi_n\left(\frac{\vartheta_1}{\sqrt{n}}, \frac{\vartheta_2}{n^{3/2}}\right) - e^{-\frac{1}{2}\Gamma\vartheta.\vartheta})d\vartheta \\ &= O(e^{-\beta n}) + \frac{1}{(2\pi)^2n^2}\int_{\{|\vartheta|\le\eta\sqrt{n}\}} e^{-iz.\vartheta}e^{-\frac{1}{2}\Gamma\vartheta.\vartheta}(e^{g(n,\vartheta)}-1)d\vartheta\,. \end{aligned} \tag{7.23}$$

Then for $0 \le r \le \eta\sqrt{n}$:

$$\begin{aligned} &\left|\int_{\{r\le|\vartheta|\le\eta\sqrt{n}\}} e^{-iz.\vartheta}e^{-\frac{1}{2}\Gamma\vartheta.\vartheta}(e^{g(n,\vartheta)}-1)d\vartheta\right| \\ &\qquad\le \int_{r\le|\vartheta|} e^{\frac{1}{2}\Gamma\vartheta.\vartheta}(1+\left|e^{g(n,\vartheta)}\right|)\,d\vartheta \\ &\qquad\le \int_{r\le|\vartheta|} e^{\frac{1}{2}\Gamma\vartheta.\vartheta}(1+e^{\frac{1}{4}\Gamma\vartheta.\vartheta})\,d\vartheta \\ &\qquad\le 2\int_{r\le|\vartheta|} e^{-\frac{1}{4}\Gamma\vartheta.\vartheta} \le 2\int_{r\le|\vartheta|} e^{-\frac{1}{4}c|\vartheta|^2} \le Ce^{-C'r^2}\,. \end{aligned} \tag{7.24}$$

We just established that there exist $\eta > 0$, $C > 0$ and $\zeta > 0$ such that for any $r \in [0, \eta\sqrt{n}]$ we have

$$p_n(k,a) - \bar{p}_n(k,a) = v_n(k,a,r) + \frac{1}{(2\pi)^2 n^2} \int_{|\vartheta| \le r} e^{-iz.\vartheta} e^{-\frac{1}{2}\Gamma\vartheta.\vartheta} (e^{g(n,\vartheta)} - 1) d\vartheta \,, \tag{7.25}$$

with

$$|v_n(k,a,r)| \le C n^{-2} e^{-\zeta r^2} \,. \tag{7.26}$$

We can now conclude, by taking $r = n^\gamma$, with $0 < \gamma < \frac{1}{2}$, since

$$\left| \int_{|\vartheta| \le r} e^{-iz.\vartheta} e^{-\frac{1}{2}\Gamma\vartheta.\vartheta} (e^{g(n,\vartheta)} - 1) d\vartheta \right| \le \\ C \int_{|\vartheta| \le r} e^{-\frac{1}{2}\Gamma\vartheta.\vartheta} (e^{C' \frac{|\vartheta|^2}{n}} - 1)\, d\vartheta \le \\ C'' \int_{|\vartheta| \le n^\gamma} \frac{|\vartheta|^2}{n} e^{-\frac{1}{2}\Gamma\vartheta.\vartheta} d\vartheta \le \frac{C''}{n} \int |\vartheta|^2 e^{-\frac{1}{2}\Gamma\vartheta.\vartheta} d\vartheta \le \frac{C'''}{n} \,. \tag{7.27}$$

If we plug this bound into (7.23), we obtain (7.11).

References

[1] Alexander, Kenneth S. Cube-root boundary fluctuations for droplets in random cluster models. *Comm. Math. Phys.* **224**, (2001), no. 3, 733–781. MR-1871907

[2] Bertoin, Jean. Levy processes. Cambridge Tracts in Mathematics, 121. *Cambridge University Press*, Cambridge, 1996. x+265 pp. MR-1406564

[3] Billingsley, Patrick. Convergence of probability measures. Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. *John Wiley and Sons, Inc.*, New York, 1999. x+277 pp.

MR-1700749

[4] Prellberg, T.; Owczarek, A.L.; Brak, R.; Guttmann, A. J. J. Finite-length scaling of collapsing directed walks. *Phys. Rev. E* **48**, (1993), 2386–2396.

[5] Caputo, Pietro; Martinelli, Fabio; Toninelli, Fabio Lucio. Mixing times of monotone surfaces and SOS interfaces: a mean curvature approach. *Comm. Math. Phys.* **311**, (2012), no. 1, 157–189. MR-2892467

[6] Caravenna, Francesco; Deuschel, Jean-Dominique. Pinning and wetting transition for $(1+1)$-dimensional fields with Laplacian interaction. *Ann. Probab.* **36**, (2008), no. 6, 2388–2433. MR-2478687

[7] Caravenna, Francesco; Deuschel, Jean-Dominique. Scaling limits of $(1+1)$-dimensional pinning models with Laplacian interaction. *Ann. Probab.* **37**, (2009), no. 3, 903–945. MR-2537545

[8] Caravenna, Francesco; Giacomin, Giambattista; Zambotti, Lorenzo. Sharp asymptotic behavior for wetting models in $(1+1)$-dimension. *Electron. J. Probab.* **11**, (2006), no. 14, 345–362 (electronic). MR-2217821

[9] Caravenna, Francesco; Sun, Rongfeng; Zygouras, Nikolaos. The continuum disordered pinning model. *Prob. Theory Rel. Fields*, (2016), no. 1-2, 17–59. arXiv:1406.5088v2 MR-3449385

[10] Carmona, Philippe; Nguyen, Gia Bao; Pétrélis, Nicolas. Interacting partially directed self-avoiding walk, from phase transition to the geometry of the collapse phase. To appear in the *Annals of Probability*, (2016), arXiv:1306.4887

[11] Dembo, Amir; Zeitouni, Ofer. Large deviations techniques and applications. Corrected reprint of the second (1998) edition. Stochastic Modelling and Applied Probability, 38. *Springer-Verlag*, Berlin, 2010. xvi+396 pp. MR-2571413

[12] Denisov, Denis; Kolb, Martin; Wachtel, Vitali. Local asymptotics for the area of random walk excursions. *J. Lond. Math. Soc. (2)* **91**, (2015), no. 2, 495–513. MR-3355112

[13] Deuschel, Jean-Dominique; Giacomin, Giambattista; Zambotti, Lorenzo. Scaling limits of equilibrium wetting models in $(1+1)$-dimension. *Probab. Theory Related Fields* **132**, (2005), no. 4, 471–500. MR-2198199

[14] Dobrushin, R.; Hryniv, O. Fluctuations of shapes of large areas under paths of random walks. *Probab. Theory Related Fields* **105**, (1996), no. 4, 423–458. MR-1402652

[15] Dobrushin, R.; Hryniv, O. Fluctuations of the phase boundary in the 2D Ising ferromagnet. *Comm. Math. Phys.* **189**, (1997), no. 2, 395–445. MR-1480026

[16] Dobrushin, R.; Kotecky, R.; Shlosman, S. Wulff construction. A global shape from local interaction. Translated from the Russian by the authors. Translations of Mathematical Monographs, 104. *American Mathematical Society*, Providence, RI, 1992. x+204 pp. MR-1181197

[17] Doney, R. A. One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* **107**, (1997), no. 4, 451–465. MR-1440141

[18] Flory, P. J. Principles of Polymer chemistry. *Cornell University Press*, Ithaca (NY), 1953.

[19] Giacomin, Giambattista. Random polymer models. *Imperial College Press*, London, 2007. xvi+242 pp. MR-2380992

[20] Guttmann, Anthony J. Analysis of series expansions for non-algebraic singularities. *J. Phys. A* **48**, (2015), no. 4, 045209, 33 pp. MR-3300262

[21] Hryniv, O.; Ioffe, D. Self-avoiding polygons: sharp asymptotics of canonical partition functions under the fixed area constraint. *Markov Process. Related Fields* **10**, (2004), no. 1, 1–64. MR-2081491

[22] Ioffe, Dmitry. Large deviations for the 2D Ising model: a lower bound without cluster expansions. *J. Statist. Phys.* **74**, (1994), no. 1-2, 411–432. MR-1257822

[23] Ioffe, Dmitry. Exact large deviation bounds up to T_c for the Ising model in two dimensions. *Probab. Theory Related Fields* **102**, (1995), no. 3, 313–330. MR-1339736

[24] Ioffe, Dmitry; Schonmann, Roberto H. Dobrushin-Kotecky-Shlosman theorem up to the critical temperature. *Comm. Math. Phys.* **199**, (1998), no. 1, 117–167. MR-1660207

[25] Janson, Svante. Brownian excursion area, Wright's constants in graph enumeration, and other Brownian areas. *Probab. Surv.* **4**, (2007), 80–145. MR-2318402

[26] Kesten, Harry. Ratio theorems for random walks. II. *J. Analyse Math.* **11**, (1963), 323–379. MR-0163365

[27] Lawler, Gregory F.; Limic, Vlada. Random walk: a modern introduction. Cambridge Studies in Advanced Mathematics, 123. *Cambridge University Press*, Cambridge, 2010. xii+364 pp. MR-2677157

[28] Le Gall, Jean-François. Itô's excursion theory and random trees. *Stochastic Process. Appl.* **120**, (2010), 721–749. MR-2603061

[29] Nguyen, Gia Bao; Pétrélis, Nicolas. A variational formula for the free energy of the partially directed polymer collapse. *J. Stat. Phys.* **151**, (2013), no. 6, 1099–1120. MR-3063498

[30] Revuz, Daniel; Yor, Marc. Continuous martingales and Brownian motion. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 293. *Springer-Verlag*, Berlin, 1991. x+533 pp. MR-1083357

[31] Serfozo, Richard. Basics of applied stochastic processes. Probability and its Applications (New York). *Springer-Verlag*, Berlin, 2009. xiv+443 pp. MR-2484222

[32] Sohier, Julien. The scaling limits of the non critical strip wetting model. *Stochastic Process. Appl.* **125**, (2015), no. 8, 3075–3103. MR-3343288

[33] Vysotsky, Vladislav. On the probability that integrated random walks stay positive. *Stochastic Process. Appl.* **120**, (2010), 1178–1193. MR-2639743

[34] R. Zwanzig; J.I. Lauritzen. Exact calculation of the partition function for a model of two dimensional polymer crystallization by chain folding. *J. Chem. Phys.* **48**, (1968), 3351.